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The Embedding of Gauged $N = 8$ Supergravity into 11 Dimensions

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Abstract

This thesis presents the complete embedding of the bosonic section of gauged $N = 8$ supergravity into 11 dimensions. The fields of 11-dimensional supergravity are reformulated in a non-linear way, such that their supersymmetry transformations can be compared to the four-dimensional ones. In this way, non-linear relations between the redefined higher-dimensional fields and the fields of $N = 8$ supergravity were already found in the literature. This is the basis for finding direct uplift Ansätze for the bosonic fields of 11-dimensional supergravity in terms of the four-dimensional ones.

This work gives the scalar Ansätze for the internal fields. First, the well known uplift formulae for the inverse metric, the three-form potential with mixed index structure and the six-form potential are summarized. Secondly, new embedding formulae for the explicit internal metric, the full three-form potential and the warp factor are presented. Additionally, two subsequent non-linear Ansätze for the full internal four-form field-strength and the Freund-Rubin term are found. Finally, the vector uplift can simply be found in terms of the obtained scalar fields.

The second part of this thesis uses the obtained embedding formulae in order to construct group invariant solutions of 11-dimensional supergravity. In such cases, the higher-dimensional fields can be written solely in terms of certain group invariant tensors that are adapted to the particular geometry of the internal space. Two such examples are discussed in detail. The first one is the well-known uplift of G_2 gauged supergravity. Furthermore, a new $\mathrm{SO}(3)\times\mathrm{SO}(3)$ invariant solution of 11-dimensional supergravity is found. In particular, the consistency of both solutions is explicitly checked for a maximally symmetric spacetime.

The results may be generalized to other compactifications, e.g. the non-compact $\mathrm{CSO}(p, q, r)$ gaugings or the reduction from type IIB supergravity to five dimensions.

Zusammenfassung

Diese Doktorarbeit behandelt die bosonische Einbettung der geeichten $N = 8$ Supergravitation in elf Dimensionen. Die höherdimensionalen Felder müssen zuerst nichtlinear umdefiniert werden, sodass ihre supersymmetrischen Transformationen mit denen der vierdimensionalen Felder verglichen werden können. So wurden in der Literatur nichtlineare Beziehungen zwischen den neu definierten elfdimensionalen Feldern und den Feldern der $N = 8$ Supergravitation gefunden. Darauf basierend können nun direkte Ansätze gefunden werden, die eine vierdimensionale in eine elfdimensionale Lösung der Supergravitation einbetten.

Die Arbeit präsentiert alle Ansätze für die skalaren internen Felder. Zuerst werden die schon bekannten Einbettungsformeln für die inverse Metrik, das Dreiform-Potential mit gemischter Indexstruktur sowie das Sechsform-Potential zusammengefasst. Danach werden neue Ansätze für die explizite interne Metrik, das vollständige Dreiform-Potential, den Warp Faktor, die Vierform Feldstärke sowie den Freund-Rubin Faktor gefunden. Die Einbettung der Vektorbosonen hängt dann nur von den skalaren Feldern ab.

Der zweite Teil der Arbeit benutzt die gefundenen Einbettungsformeln, um gruppeninvariante Lösungen der elfdimensionalen Supergravitation zu finden. In solchen Fällen hängen die höherdimensionalen Felder ausschließlich von speziellen gruppeninvarianten Tensoren ab, die auf die jeweilige interne Geometrie angepasst sind. Als Beispiel wird zuerst die schon bekannte Einbettung der G_2 invarianten Supergravitation zusammengefasst. Dann wird eine neue $SO(3)\times SO(3)$ invariante Lösung der elfdimensionalen Supergravitation gefunden. Schließlich wird die Konsistenz der gefundenen Lösungen für eine maximal symmetrische Raumzeit überprüft.

Die Ergebnisse können auf andere Kompaktifizierungen verallgemeinert werden, z.B. auf die nichtkompakten $CSO(p, q, r)$ Eichungen oder auf die Reduzierung der Typ IIB Supergravitation zu fünf Dimensionen.

Contents

1. Introduction

1.1. General Relativity and the Standard Model of Particle Physics

The four fundamental forces that are observed in nature are well described by two different physical theories — general relativity and the Standard Model of particle physics. On the one hand, Einstein's general relativity describes gravity as the result of curved four-dimensional spacetime [1, 2, 3, 4]. This classical field theory is based on the Einstein-Hilbert action, whose Euler-Lagrange equations (the 'Einstein equations') set restrictions on the spacetime itself. Mathematically, these relations are second-order differential equations for the metric, which is a symmetric 4×4 matrix that fully describes the spacetime.

This theory of gravity passed several non-trivial tests during the past 100 years. For example, solving the Einstein equations assuming spherical symmetry leads to the celebrated Schwarzschild metric [5], which specifies the spacetime around stars and black holes. This precisely describes the movement of massive test particles and explains the bending of light rays in a gravitational field. A further solution of the linearized Einstein equations describes gravitational waves, which were only recently discovered [6]. Finally, general relativity allows us to study cosmology: Assuming an isotropic and homogeneous spacetime at very large scales, one finds the dynamics of the universe itself. In particular, this dynamics is determined by the Friedmann equations, which are the reduced Einstein equations in that case [7, 8]. This model provides a possible explanation for the observed expanding universe and also gives information about its size and age.

Unfortunately, general relativity does not explain all the observed gravitational effects. For example, it leads to the wrong radial dependence of the rotational velocities of stars within a galaxy. In order to solve this open problem, there are two common suggestions: One is the modification of the gravitational laws themselves on large scales, described for example by Tensor-Vector-Scalar (TeVeS) or $f(R)$ theories [9, 10, 11]. Since none of the existing modifications consistently describe all astronomical observations, another idea is to assume 'dark matter' within galaxies. These are massive particles that affect the gravitational interactions but are otherwise almost invisible. The search for such particles is a current field of research but so far, none have been found. A second problem of general relativity is related to the expansion of the universe. In principle, the expansion requires a vacuum energy that acts against the gravitational collapse of the universe, which is commonly known as dark energy. The notion of a vacuum energy is also well known in particle physics, but the estimated values of both theories differ by 120 orders of magnitudes. This discrepancy is one of the most puzzling problems in physics.

On the other hand, the Standard Model of particle physics describes the electromagnetic, weak and strong forces of nature [12, 13]. It is a 'quantum' field theory that is based on a complicated action that includes all elementary particle fields (such as quarks, leptons and gauge bosons). The main difference to a classical field theory is in the description of interactions. A classical field configuration precisely satisfies the Euler-Lagrange equations and therefore extremizes the associated action. In contrast, all information that one can seek about a quantum field configuration is its *probability* to be in a certain state. Mathematically, this probability is determined by the exponential of the action — the smaller the value of the action for a certain field configuration, the higher is the corresponding probability. One may finally compute the *physical observable expectation value* for a quantum field. In particular, in the classical limit, this field expectation value coincides with the classical field configuration that extremizes the action.

There are only few restrictions to construct the most general action for the Standard Model of particle physics. In principle, these rules determine the corresponding interaction probabilities. For example, the action must be invariant under the Lorentz group SO(1,3) and the 'gauge group' U(1)×SU(2)×SU(3). Here, the strong sector is described by an SU(3) gauge group of 'color' and the corresponding interactions must be described using non-perturbative methods. On the other hand, the gauge group U(1)×SU(2) describes the electroweak sector, in which all calculations may be performed perturbatively using Feynman integrals and renormalization methods. However, Lorentz and U(1)×SU(2) gauge invariance lead to a serious problem, namely that all particles must be massless. The simplest solution was suggested by Higgs, Englert, Brout, Guralnik, Hagen and Kibble in 1964 [14, 15, 16]. They proposed a mechanism that breaks the U(1)×SU(2) symmetry group down to the U(1) gauge group of electromagnetism. This provides the correct masses for the W and Z gauge bosons of the weak force, whereas the photon of the electromagnetic force remains massless. Furthermore, leptons may acquire a mass via the introduction of Yukawa couplings. This spontaneous symmetry breaking only requires one additional *scalar particle — the Higgs boson*, which was finally found four years ago at the Large Hadron Collider at CERN [17, 18].

Although tested in diverse experiments, the Standard Model can not explain all observed electroweak and strong effects. The first one is related to the neutrino flavor oscillations: Experiments in the 1960s measured the flux of solar neutrinos through the earth. The number of detected electron neutrinos was far too small compared to the expected number (based on the estimated nuclear fusion reactions in the sun). This is called the 'solar neutrino problem'. In 1968, Pontecorvo proposed that *massive* neutrinos could change their flavor when traveling over long distances [19]. Finally in 1998 and 2001, the Super-Kamiokande Observatory and the Sudbury Neutrino observatory independently measured *all* solar neutrinos traveling through the earth and found that the missing amount of electron neutrinos was compensated by the observed amount of muon and tau neutrinos [20, 21]. This proved that neutrinos indeed oscillate in flavor and are hence not all massless, which contradicts the prediction of the Standard Model.

A second serious puzzle is the 'hierarchy problem': Naturally, the quantum corrections to the renormalized mass of the Higgs boson would make it far heavier than observed in current detectors. Assuming that the Standard Model of particle physics is complete, the only reason for the relatively small Higgs mass would be a precise 'fine-tuning' of the 'bare Higgs mass' — such that it exactly cancels the quantum corrections. Since this seems to be very unnatural for physicists, it is strongly believed that physics beyond the Standard Model does not require such a fine-tuning. The hierarchy problem was one of the main motivations to study supersymmetry — a quantum field theory in which each elementary particle has a supersymmetric partner with opposite spin statistics. In this case, the quantum corrections to the Higgs mass coming from a certain particle and its super-partner precisely cancel. Hence, the bare Higgs mass must not be unnaturally fine-tuned. Unfortunately, no supersymmetric partner of an elementary particle has been observed so far.

All in all, both theories can not describe the respective opposite regime: General relativity is a classical field theory, and a quantum field theory including curved spacetime would inevitably result in divergences that can not be renormalized. It is therefore only natural to ask for a more fundamental theory that unifies *all* forces of nature. Such a 'theory of everything' could then solve the current open problems that exist within the limits of both theories.

1.2. Kaluza-Klein Theory and Supergravity

One attempt to unify Einstein's general relativity and the Standard Model of particle physics is Kaluza-Klein theory. The original idea was to construct a pure $D > 4$ gravity theory based on a D-dimensional Einstein-Hilbert action. However, our world is four-dimensional, so, where are the extra dimensions? The answer is given by a mathematical concept called dimensional reduction: The extra dimensions compactify on such a small scale that one does not observe them in everyday's life. The result is a four-dimensional curved spacetime and the compactified dimensions lead to internal gauge symmetries.

Let us consider the initial example that inspired Kaluza and Klein to propose this theory in the 1920s [22, 23]: They considered a five-dimensional theory of gravity with an Einstein-Hilbert action for the five-dimensional metric. At each point p in spacetime, the fifth axis now becomes isomorphic to a circle when gluing the five-dimensional points $(p, -\infty)$ and $(p, +\infty)$ together. This procedure is called U(1) compactification, because the isometry group of a circle is U(1). The resulting theory is hence, general relativity describing the spacetime together with an internal U(1) gauge symmetry of electromagnetism. Under this compactification, the 15 degrees of freedom of the five-dimensional metric (a symmetric 5×5 matrix) split as follows: 10 components describe the four-dimensional spacetime metric (a symmetric 4×4 matrix), one four-vector describes the photon and the remaining degree of freedom is contained in a real scalar field. Using this decomposition, the five-dimensional Einstein-Hilbert action consistently reduces to a four-dimensional Einstein-Hilbert term plus several other terms, which describe the interactions between the photon and the scalar field. In particular, the Euler-Lagrange equations reduce to the Einstein equations for the spacetime, the Maxwell equations of electrodynamics and an additional relation for the scalar field. Unfortunately, this procedure does not lead to the correct electromagnetic interactions that are observed in particle detectors.

Later, physicists extended Kaluza-Klein theory to obtain larger gauge groups, e.g. to find the Standard Model of particle physics in the compactified dimensions [24, 25, 26, 27, 28, 29]. In the general approach, the manifold $\mathcal{M}_D$ of a D-dimensional gravity theory spontaneously compactifies according to

$$\mathcal{M}_D = \mathcal{M}_4 \times \mathcal{M}_{D-4}. \tag{1.1}$$

Let us assume for a moment that a stable ground state solution is already known, i.e. a complete set of D-dimensional fields $\Phi_0(x, y)$ that fulfill the corresponding equations of motion. Here, x and y denote the coordinates on $\mathcal{M}_4$ and $\mathcal{M}_{D-4}$ respectively. Further solutions

$$\Phi(x, y) = \Phi_0(x, y) + \sum_n \Phi^{(n)}(x) Y^{(n)}(y) \tag{1.2}$$

can then be found by a linear expansion of the higher-dimensional fields around this ground state. In particular, the expansion coefficients $\Phi^{(n)}(x)$ represent the physical fields of a four-dimensional theory and $Y^{(n)}(y)$ are the eigenfunctions of a mass operator that acts on the internal manifold $\mathcal{M}_{D-4}$. This gives a finite number of massless Kaluza-Klein modes — the Yang Mills fields — and infinitely many massive states. Since these massive modes would be too heavy to detect in current observations, one discards them and restricts to the zero mass sector. This *truncation* of the theory is called the 'low energy limit' and must be *consistent*, i.e. the fields $\Phi(x, y)$ must still satisfy the 11-dimensional equations of motion. Only when this is the case, each four-dimensional solution $\{\Phi^{(n)}(x)\}$ finally corresponds to a solution $\{\Phi(x, y)\}$ of the D-dimensional gravity theory. In particular, this method is a powerful tool to understand a complicated four-dimensional theory in a higher-dimensional framework — the complexity is induced by the reduction scheme.

The idea of relating gravity theories in different dimensions to each other finally led to a more general attempt to unify general relativity and the Standard Model of particle physics: *Supergravity*. In principle, the basic idea was to make supersymmetry local [30]. Later, a first consistent gravity theory with one supersymmetry was proposed in 1976 — the so-called minimal supergravity in four dimensions [31]. In the following years, several generalizations were suggested, e.g. $N = 8$ supergravity in four dimensions. It has the maximal number of supersymmetries, assuming that there is no particle with spin higher than two. It was first investigated by de Wit and Freedman in 1977 [32] but turned out to be difficult to construct. A second example is 11-dimensional supergravity, which was proposed by Cremmer and Julia only one year later [33]. In these years, the uniqueness of the latter theory made it an adequate candidate for a 'theory of everything'. Indeed, 11 is the maximal dimension consistent with a single graviton of spin two [34] and the minimal dimension to cover the Standard Model of particle physics under a compactification of the extra dimensions [29]. Unfortunately, the corresponding reductions turned out to be *inconsistent*.

In general, to establish a *consistent* Kaluza-Klein reduction from one supergravity theory to another is even more complicated than in the non-supersymmetric case [35]. The reason is the following: For the above *linear* expansion in Eq. (1.2), the supersymmetry transformations in both theories must be related via

$$\delta\Phi(x,y) = \sum_n \delta\Phi^{(n)}(x) Y^{(n)}(y). \tag{1.3}$$

However, the supersymmetry transformations are *non-linear* in all fields. Hence, the lhs is equal to a non-linear combination of the higher-dimensional fields, $\tilde{F}(\Phi(x,y))$. In particular, when applying Eq. (1.2) on the lhs again, the y-dependent factor also becomes non-linear in the eigenfunctions $Y^{(n)}(y)$. This is a contradiction to Eq. (1.3) and hence, the linear expansion in Eq. (1.2) and its truncation can not be consistent. Therefore, the only way out is a non-linear modification of Eq. (1.2), i.e. the higher-dimensional fields must be *redefined in a non-linear way*, $\Phi(x,y) \to F(\Phi(x,y))$, such that both,

$$\begin{aligned} F(\Phi(x,y)) &= F(\Phi_0(x,y)) + \sum_n \Phi^{(n)}(x) Y^{(n)}(y), \\ \delta F(\Phi(x,y)) &= \sum_n \delta\Phi^{(n)}(x) Y^{(n)}(y) \end{aligned} \tag{1.4}$$

hold separately. If such a non-linear modification exists, the truncation is automatically consistent, because the supersymmetry transformations close on-shell. In other words, if $\{\Phi^{(n)}(x)\}$ is a classical four-dimensional solution and the supersymmetry transformations satisfy Eq. (1.4), then, the fields $\Phi(x,y)$ also satisfy the 11-dimensional equations of motion.

The first investigated consistent compactification of 11-dimensional supergravity is the reduction of the extra dimensions on a seven-torus. The resulting four-dimensional theory is *ungauged* $N = 8$ supergravity [36][1]. Here, the fermions transform under a local

[1]Note that only in this way, the (relatively complicated) $N = 8$ supergravity could finally be completed using the scheme of dimensional reduction.

SU(8) gauge group and the corresponding scalar and vector fields transform under a global E_7 group of 'duality invariance'. In particular, the four-dimensional supersymmetry transformations are manifestly SU(8) and E_7 covariant. This is the basis to find the redefinitions of the 11-dimensional fields, $\Phi(x,y) \to F(\Phi(x,y))$. One requires that the resulting supersymmetry transformations are also SU(8) and E_7 covariant. In this way, the complete torus reduction can consistently be found.

On the other hand, 11-dimensional supergravity may be reduced to *maximally gauged* $N = 8$ supergravity [37], when the internal dimensions are compactified on a seven-sphere [38, 39, 40, 41]. This can be done by gluing together all points at infinity of the seven extra dimensions. Indeed, the internal space is then isomorphic to a seven-sphere and the corresponding isometry group is SO(8) — the maximal gauge group that is allowed in $N = 8$ supergravity. Furthermore, when deforming the seven-sphere in a certain way, one finds other (non-maximal) gaugings, such as G_2, SO(3)×SO(3) or SU(3)×U(1)×U(1) invariant supergravity. Unfortunately, all these compact gaugings break the global E_7 group of duality invariance in $N = 8$ supergravity. However, using the 'embedding tensor formalism' [42], one may extend the electric vector fields by magnetic duals, such that the supersymmetry transformations can still be written in an SU(8) and E_7 covariant way. Therefore, the 11-dimensional non-linear field redefinitions can still be found using the guideline of SU(8) and E_7 covariance. In particular, all these Kaluza-Klein compactifications on a deformed seven-sphere are *consistent*.

Right after the development of supergravity up until now, there has been active research in the *embedding* of gauged $N = 8$ supergravity into 11-dimensional supergravity [41, 43, 44, 45, 46, 47]: One constructs a particular 11-dimensional solution, which consistently reduces to a given solution of $N = 8$ supergravity under a Kaluza-Klein compactification. The main task in establishing such a program is to find explicit *uplift Ansätze* for the 11-dimensional fields $\Phi(x,y)$ in terms of the four-dimensional ones $\Phi^{(n)}(x)$, starting from the non-linear relations in Eq. (1.4). In particular, the consistency of the reduction implies that the constructed fields $\Phi(x,y)$ automatically satisfy the higher-dimensional equations of motion as long as $\{\Phi^{(n)}(x)\}$ is a solution of $N = 8$ supergravity. The embedding formalism can hence be seen as a tool to find new 11-dimensional supergravity solutions.

1.3. Thesis Aim and Own Results

This thesis presents the full *bosonic* embedding of gauged $N = 8$ supergravity into 11 dimensions. A first formula for the inverse internal metric has already been found in Ref. [43]. For certain gaugings, the resulting expression could then be inverted to find the internal metric. Furthermore, the non-linear Ansätze for the internal form potentials have been found in Refs. [45, 47, 48], which also required the explicit metric expression. This thesis derives a new *direct* uplift Ansatz for the internal metric [49, 50]. In particular, the inversion procedure becomes redundant and all scalar uplifts complete in general. Finally, two more subsequent Ansätze for the internal 'four-form field-strength' and the 'Freund-Rubin term' are found [48, 49, 50]. In particular, these are required in order to explicitly check the consistency of certain group invariant solutions of 11-dimensional supergravity. The derived non-linear Ansätze for the scalar fields imply simple embedding formulae for the higher-dimensional vectors. Hence, the complete bosonic solution of 11-dimensional supergravity can be found.

Furthermore, assuming the spacetime of the compactification in Eq. (1.1) to be maximally symmetric yields even simpler solutions, namely the 'Freund-Rubin solutions with flux'. Here, also the vectors vanish and the presented Ansätze are already *sufficient* to construct *all* required fields. However, this simplification is only consistent, when the obtained 11-dimensional fields are evaluated at a stationary point of the scalar potential. As special cases, this thesis presents the G_2 and $SO(3)\times SO(3)$ invariant Freund-Rubin solutions of 11-dimensional supergravity and explicitly checks that the obtained fields satisfy the corresponding equations of motion. The derivation of the G_2 invariant solution is a summary of known results together with a new expression for the internal six-form potential. The $SO(3)\times SO(3)$ invariant solution is completely new [51].

The outline of the manuscript is as follows: The first chapter introduces the gauged $N = 8$ supergravity in four dimensions. It presents the fields and their supersymmetry transformations as well as some E_7 properties of the scalar fields of this theory. Chapter 3 is devoted to the fields and supersymmetry transformations of 11-dimensional supergravity. In particular, the non-linear SU(8) and E_7 field reformulations are discussed in detail. Chapter 4 then deals with the embedding of gauged $N = 8$ supergravity into 11 dimensions. Based on the consistent relation between the redefined fields and the four-dimensional ones, it presents the explicit non-linear uplift Ansätze for the 11-dimensional scalar fields.

The second part of this thesis starts with Chapter 5. It gives a general overview to the application of the derived uplift Ansätze for certain gaugings of the $N = 8$ supergravity. In particular, it shows how the higher-dimensional fields can be written in terms of certain group invariant tensors, which brings the solution in a simpler, manageable form. This forms the basis for the explicit examples in Chapters 6 and 7, which construct the G_2 and $SO(3)\times SO(3)$ invariant Freund-Rubin solutions of 11-dimensional supergravity. In particular, they also show explicitly that the obtained fields satisfy the reduced equations of motion in that case. Chapter 8 finally concludes this thesis.

Appendix A gives some useful identities for the Γ matrices and Killing forms that appear within the S^7 reduction. A simplification for the $\mathcal{C}$ tensor that occurs in the three-form Ansatz is presented in Appendix B. Finally, Appendix C derives all 'S^7 tensor identities', which are necessary to bring the $SO(3)\times SO(3)$ invariant solution of 11-dimensional supergravity into a manageable form.

2. $N = 8$ Supergravity

This first chapter is devoted to the $N = 8$ supergravity theory [32, 36, 37]. Of all supergravity theories, $N = 8$ is the maximal number of supersymmetries without introducing spin fields higher than two. The local gauge group is SO(8)×SU(8) and the latter may be extended to a global group of duality invariance. As it turns out, this global group is well described by an exceptional E_7 symmetry.

The first part of this section presents the fields of $N = 8$ supergravity and explains the emergence of the duality symmetry in detail. Then, the second part gives the corresponding supersymmetry transformations, which are written in an SU(8) and E_7 covariant way. Finally, Section 2.3 gives some general E_7 properties that restrict the scalar fields of $N = 8$ supergravity.

2.1. The Fields of $N = 8$ Supergravity

The field content is described by an irreducible $N = 8$ super-multiplet. This decomposes into a spin 2 graviton, eight spin 3/2 Rarita-Schwinger spinors, 28 gauge bosons, 56 Majorana spin 1/2 fermions as well as 35 + 35 scalar and pseudo-scalar fields. In the following, these fields are presented step by step.

First, the graviton is described by the metric of the four-dimensional spacetime, $\mathring{g}_{\mu\nu}(x)$, in a local coordinate chart x^μ. Using the Cartan formalism, one introduces a corresponding vierbein $\mathring{e}_\mu{}^\alpha(x)$, such that

$$\mathring{g}_{\mu\nu} = \delta_{\alpha\beta}\mathring{e}_\mu{}^\alpha \mathring{e}_\nu{}^\beta. \tag{2.1}$$

Here, $\delta_{\alpha\beta}$ denotes the flat euclidean tangent space metric — the time coordinate is imaginary. Spacetime and tangent space indices are denoted by Greek letters. In general, the rule of thumb is: Letters from the middle of an alphabet always denote curved spacetime indices and letters from the beginning are the corresponding tangent space indices. The vierbein (or metric) is a singlet under the gauge group and as usual, it defines the Riemann and Ricci curvature tensors $\mathring{R}^\mu{}_{\nu\rho\sigma}(x)$, $\mathring{R}_{\mu\nu}(x) = \mathring{R}^\rho{}_{\mu\rho\nu}(x)$ as well as the Ricci scalar $\mathring{R}(x) = g^{\mu\nu}(x)\mathring{R}_{\mu\nu}(x)$. Finally, the four-dimensional volume form is

$$\mathrm{d}V = \sqrt{|\mathring{g}|}\mathrm{d}^4x,$$

where $\mathring{g}$ denotes the determinant of the metric. It is related to the determinant of the vierbein via $\mathring{g} = \mathring{e}^2$. Thus, the corresponding Einstein-Hilbert action reads

$$S = -\frac{1}{2} \int \mathring{e} \mathring{R} \, \mathrm{d}^4 x, \tag{2.2}$$

where the normalization factor is consistent with the conventions of Ref. [37].

Secondly, let us discuss the fermions. The eight Rarita-Schwinger spin 3/2 fields are denoted by a spinor ${\phi_\mu}^i(x)$[1]. It belongs to the irreducible eight-dimensional representation of the chiral SU(8) gauge group. The corresponding SU(8) index i runs from 1 to 8 and can be raised or lowered by complex conjugation. An upper SU(8) index corresponds to positive and a lower one to negative chirality. The four spinorial components of ${\phi_\mu}^i(x)$ are labeled by indices $\hat{\alpha}, \hat{\beta}, \ldots = 1, \ldots 4$, which are most often suppressed,

$$ {\phi_\mu}^i = {(\phi_\mu)_{\hat{\alpha}}}^i. $$

The corresponding four-dimensional matrices $\gamma_\alpha = (\gamma_\alpha)_{\hat{\alpha}\hat{\beta}}$ satisfy the Clifford algebra

$$\{\gamma_\alpha, \gamma_\beta\} = 2\delta_{\alpha\beta} \mathbb{I}_{4\times 4}. \tag{2.3}$$

They also define the fifth γ matrix as $\gamma_5 = \gamma_0\gamma_1\gamma_2\gamma_3$.

The 56 spin 1/2 fermions are Majorana spinors $\chi^{ijk}(x)$ with fully antisymmetric chiral SU(8) indices $[ijk]$. They belong to the chiral **56** representation of SU(8). Again, the spinorial index $\hat{\alpha}$ is suppressed,

$$\chi^{ijk} = {\chi_{\hat{\alpha}}}^{ijk},$$

and the SU(8) indices are raised and lowered by complex conjugation.

Furthermore, there are 70 scalar degrees of freedom in $N = 8$ supergravity. Mathematically, these scalar fields extend the local SU(8) gauge group to the group G of duality invariance [37]. Since SU(8) is 63-dimensional, the group G must have $70 + 63 = 133$ generators. This counting argument quite naturally led to the conjecture, that G is the exceptional group E_7 of Killing-Cartan [36], whose maximal compact subgroup is indeed, SU(8). To be more precise, one must choose the non-compact version $\mathrm{E}_{7(7)}$. Accordingly, the scalars are parametrized by a 56-bein $\mathcal{V}(x)$, which belongs to the fundamental **56** representation of $\mathrm{E}_{7(7)}$. This 56-bein transforms under local SU(8) and global $\mathrm{E}_{7(7)}$ rotations as[2]

$$\mathcal{V}(x) \to U(x)\mathcal{V}(x)E^{-1}, \qquad U(x) \in \mathrm{SU}(8), \qquad E \in \mathrm{E}_{7(7)}.$$

Hence, the 133 scalar degrees of freedom in the 56-bein are not independent and fall into equivalence classes of SU(8). The remainder are the 70 scalars of $N = 8$ supergravity, which parametrize the *coset space* $\mathrm{E}_{7(7)}/\mathrm{SU}(8)$.

[1] In Refs. [36, 37], this spinor was denoted by ${\psi_\mu}^i$.

[2] This is in full analogy to the vierbein ${\mathring{e}_\mu}^\alpha$, which transforms under global GL(4,$\mathbb{R}$) diffeomorphisms from the left and under local SO(1,3) rotations from the right.

There are two usual representations of the 56-bein. The first is via a decomposition into 28×28 sub-matrices $u_{ij}{}^{IJ}(x)$ and $v_{ij\,IJ}(x)$,

$$\mathcal{V} = \begin{pmatrix} u_{ij}{}^{IJ} & v_{ij\,IJ} \\ v^{ij\,IJ} & u^{ij}{}_{IJ} \end{pmatrix} . \tag{2.4}$$

Capital indices refer to the SL(8,$\mathbb{R}$) decomposition of $\mathrm{E}_{7(7)}$,

$$\mathbf{56} \to \mathbf{28} \oplus \overline{\mathbf{28}}, \tag{2.5}$$

and lower case indices are antisymmetric bi-vector indices of the chiral SU(8). Hence, the latter are raised and lowered via complex conjugation,

$$u^{ij}{}_{IJ} = \left(u_{ij}{}^{IJ}\right)^* , \qquad v^{ij\,IJ} = \left(v_{ij\,IJ}\right)^* . \tag{2.6}$$

A second parametrization is given in another gauge, [52]

$$\hat{\mathcal{V}} = \begin{pmatrix} \hat{\mathcal{V}}^{IJ}{}_{ij} & \hat{\mathcal{V}}_{IJ\,ij} \\ \hat{\mathcal{V}}^{IJ\,ij} & \hat{\mathcal{V}}_{IJ}{}^{ij} \end{pmatrix} , \tag{2.7}$$

which is related to $\mathcal{V}$ via

$$\hat{\mathcal{V}}^{IJ}{}_{ij} = \frac{i}{\sqrt{2}} \left(u_{ij}{}^{IJ} + v_{ij\,IJ}\right) , \qquad \hat{\mathcal{V}}_{IJ\,ij} = -\frac{1}{\sqrt{2}} \left(u_{ij}{}^{IJ} - v_{ij\,IJ}\right) , \tag{2.8}$$

$$\hat{\mathcal{V}}^{IJ\,ij} = -\frac{i}{\sqrt{2}} \left(u^{ij}{}_{IJ} + v^{ij\,IJ}\right) , \qquad \hat{\mathcal{V}}_{IJ}{}^{ij} = -\frac{1}{\sqrt{2}} \left(u^{ij}{}_{IJ} - v^{ij\,IJ}\right) . \tag{2.9}$$

Again, capital indices in Eq. (2.7) refer to the SL(8,$\mathbb{R}$) decomposition of the **56** representation of $\mathrm{E}_{7(7)}$. In this case, it is more convenient to write the 56-bein as a $\mathrm{E}_{7(7)}$ vector

$$\hat{\mathcal{V}}^{\mathcal{M}}{}_{ij} = \left(\hat{\mathcal{V}}^{IJ}{}_{ij}, \quad \hat{\mathcal{V}}_{IJ\,ij}\right) .$$

The 56-dimensional index $\mathcal{M}$ belongs to the **56** representation of $\mathrm{E}_{7(7)}$ and can be raised and lowered with a certain symplectic form $\Omega_{\mathcal{M}\mathcal{N}}$. This is explained in full detail in Section 2.3. Finally, one notes that the second row in Eq. (2.7) is the complex conjugate of the first one. Written in an $\mathrm{E}_{7(7)}$ covariant way, this means

$$\hat{\mathcal{V}}^{\mathcal{M}\,ij} = \left(\hat{\mathcal{V}}^{\mathcal{M}}{}_{ij}\right)^* = \left(\hat{\mathcal{V}}^{IJ\,ij}, \quad \hat{\mathcal{V}}_{IJ}{}^{ij}\right) . \tag{2.10}$$

With the above definition of the 56-bein, the scalar potential $V(x)$ of $N = 8$ supergravity is given in terms of the 'T tensor'

$$T_i^{jkl}(x) = \left(u^{kl}{}_{IJ} + v^{kl\,IJ}\right) \left(u_{im}{}^{JK} u^{jm}{}_{KI} - v_{im\,JK} v^{jm\,KI}\right)(x), \tag{2.11}$$

which satisfies the convenient property

$$\left(u_{pq}{}^{IJ} + v_{pq\,IJ}\right)\left(u^{ij}{}_{IK}v^{kl\,JK} - v^{ij\,IK}u^{kl}{}_{JK}\right) = \frac{4}{3}\delta^{[i}{}_{[p}T_{q]}{}^{jkl]}. \tag{2.12}$$

In particular, [37, 41]

$$V = \frac{1}{24}g^2 A_{2\,i}{}^{jkl}A_2{}^i{}_{jkl} - \frac{3}{4}g^2 A_1^{ij}A_{1ij}, \tag{2.13}$$

where g is a coupling constant and the A_1 and A_2 tensors are given by

$$A_1^{ij} = \frac{4}{21}T_k{}^{ikj}, \qquad A_{2\,i}{}^{jkl} = -\frac{4}{3}T_i{}^{[jkl]}. \tag{2.14}$$

Finally, the 28 gauge fields $A_\mu{}^{IJ}(x)$ belong to the adjoint **28** representation of the SO(8) symmetry group[3] and transform as a singlet under SU(8). This gauge group breaks the global E_7 group of duality invariance. However, the 28 'electric' vector fields can be extended by 28 'magnetic' duals $A_{\mu\,IJ}(x)$ in the 'embedding tensor formalism' [42]. Only both, electric and magnetic vector fields together then constitute an irreducible **56** representation of $E_{7(7)}$,

$$A_\mu{}^{\mathcal{M}} = \left(A_\mu{}^{IJ}, \quad A_{\mu\,IJ}\right).$$

This represents the corresponding SL(8,$\mathbb{R}$) decomposition given by Eq. (2.5).

The most general resulting Lagrangian is given in Refs. [32, 36, 37]. Here, it is not required for the embedding of $N = 8$ supergravity into 11 dimensions.

2.2. Supersymmetry Transformations

With the above definitions of the fields, the corresponding supersymmetry transformations read [37, 41, 42][4]

$$\delta_\epsilon \mathring{e}_\mu{}^\alpha = \frac{1}{2}\bar{\epsilon}^i\gamma^\alpha\phi_{\mu\,i} + \text{h.c.}, \tag{2.15}$$

$$\delta_\epsilon u_{ij}{}^{IJ} = -\sqrt{2}\Sigma_{ijkl}(\epsilon)v^{kl\,IJ}, \qquad \delta_\epsilon A_\mu{}^{IJ} = -\frac{1}{2}X_\mu{}^{ij}(\epsilon)\left(u_{ij}{}^{IJ} + v_{ij\,IJ}\right) + \text{h.c.}, \tag{2.16}$$

$$\delta_\epsilon v_{ij\,IJ} = -\sqrt{2}\Sigma_{ijkl}(\epsilon)u^{kl}{}_{IJ}, \qquad \delta_\epsilon A_{\mu\,IJ} = -\frac{i}{2}X_\mu{}^{ij}(\epsilon)\left(u_{ij}{}^{IJ} - v_{ij\,IJ}\right) + \text{h.c.}, \tag{2.17}$$

with

$$X_\mu{}^{ij}(\epsilon) = 2\sqrt{2}\bar{\epsilon}^i\phi_\mu{}^j + \bar{\epsilon}_k\mathring{\gamma}_\mu\chi^{ijk}, \tag{2.18}$$

$$\Sigma_{ijkl}(\epsilon) = \bar{\epsilon}_{[i}\chi_{jkl]} + \frac{1}{4!}\epsilon_{ijklmnpq}\bar{\epsilon}^m\chi^{npq}. \tag{2.19}$$

[3]The antisymmetric bi-vector indices $[IJ]$ belong to SO(8).

[4]The conventions of Ref. [41] are used. The respective ϵ differs from the one in Ref. [37] by a factor of 1/2.

Here, $\Sigma_{ijkl}(\epsilon)$ is complex selfdual, the second term in Eq. (2.19) is the complex Hodge dual of the first term and the ϵ tensor is the corresponding totally anti-symmetric tensor with chiral SU(8) indices. In all the above relations, the transformation parameter is a four-spinor ϵ^i given in the fundamental **8** representation of the chiral SU(8) symmetry group. It also carries a suppressed spinor index $\hat{\alpha}$, e.g. the first supersymmetry transformation relation in Eq. (2.15) actually reads

$$\delta_\epsilon \mathring{e}_\mu{}^\alpha = (\bar{\epsilon})_{\hat{\alpha}}{}^i (\gamma^\alpha)_{\hat{\alpha}\hat{\beta}} (\phi_\mu)_{\hat{\beta}\, i} + \text{h.c.}$$

One may now write the above supersymmetry transformations for the scalar and vector fields in a manifestly $E_{7(7)}$ covariant way,

$$\delta_\epsilon \mathcal{V} = -\sqrt{2} \begin{pmatrix} 0 & \Sigma(\epsilon) \\ \Sigma^*(\epsilon) & 0 \end{pmatrix} \mathcal{V}, \tag{2.20}$$

$$\delta_\epsilon \hat{\mathcal{V}}^{\mathcal{M}}{}_{ij} = \sqrt{2} \Sigma_{ijkl}(\epsilon) \hat{\mathcal{V}}^{\mathcal{M}\, kl}, \tag{2.21}$$

$$\delta_\epsilon A_\mu{}^{\mathcal{M}} = \frac{i}{\sqrt{2}} X_\mu{}^{ij}(\epsilon) \hat{\mathcal{V}}^{\mathcal{M}}{}_{ij} + \text{h.c.}, \tag{2.22}$$

where the first equation is equivalent to

$$\delta_\epsilon \begin{pmatrix} u_{ij}{}^{IJ} & v_{ij\, IJ} \\ v^{ij\, IJ} & u^{ij}{}_{IJ} \end{pmatrix} = -\sqrt{2} \begin{pmatrix} 0 & \Sigma_{ijkl} \\ \Sigma^{ijkl} & 0 \end{pmatrix} \begin{pmatrix} u_{kl}{}^{IJ} & v_{kl\, IJ} \\ v^{kl\, IJ} & u^{kl}{}_{IJ} \end{pmatrix}. \tag{2.23}$$

As a final remark, the supersymmetry transformation for the scalar potential is given by [53]

$$\delta_\epsilon V = \frac{\sqrt{2}}{24} g^2 \Sigma_{ijkl}(\epsilon) Q^{ijkl} + \text{h.c.}, \tag{2.24}$$

where the Q tensor is defined in terms of the T tensor,

$$Q^{ijkl} = \frac{3}{4} A_{2\, m}{}^{n[ij} A_{2\, n}{}^{kl]m} - A_1{}^{m[i} A_{2\, m}{}^{jkl]}. \tag{2.25}$$

Note that the expression on the rhs of Eq. (2.24) must vanish at the stationary points of $V(x)$. In particular, since $\Sigma_{ijkl}(\epsilon)$ is complex selfdual, $Q^{ijkl}(x)$ must be complex anti-selfdual at stationary points of the potential.

In Chapter 4, the supersymmetry transformations for the vierbein, the scalar and vector fields in Eqs. (2.15, 2.21, 2.22) will be considered to the respective 11-dimensional transformations. For the embedding of $N = 8$ supergravity into 11 dimensions, this comparison forms the basis for finding the correct bosonic uplift relations between the lower- and higher-dimensional fields. Since this thesis does not discuss the fermionic uplift, the corresponding supersymmetry transformation laws for $\phi_\mu{}^i$ and χ^{ijk} [37, 41] are not listed here.

2.3. Some E_7 Properties of the Four-Dimensional Scalars

This section collects all important properties for the 56-beine $\mathcal{V}(x)$ and $\hat{\mathcal{V}}(x)$ as well as for the scalar fields $u_{ij}{}^{IJ}(x)$ and $v_{ij\,IJ}(x)$ presented above. The derivation of these properties is based on the $E_{7(7)}$ group of duality transformations.

First, the inverse of the 56-bein $\mathcal{V}(x)$ is given by [37]

$$\mathcal{V}^{-1} = \begin{pmatrix} u^{ij}{}_{IJ} & -v_{ij\,IJ} \\ -v^{ij\,IJ} & u_{ij}{}^{IJ} \end{pmatrix}. \tag{2.26}$$

In combination with Eq. (2.4), this gives the well-known identities for the scalar fields [37]

$$u^{ij}{}_{IJ} u_{kl}{}^{IJ} - v^{ij\,IJ} v_{kl\,IJ} = \delta^{ij}_{kl}, \tag{2.27}$$
$$u^{ij}{}_{IJ} v^{kl\,IJ} - v^{ij\,IJ} u^{kl}{}_{IJ} = 0, \tag{2.28}$$
$$u^{ij}{}_{IJ} u_{ij}{}^{KL} - v_{ij\,IJ} v^{ij\,KL} = \delta^{IJ}_{KL}, \tag{2.29}$$
$$u^{ij}{}_{IJ} v_{ij\,KL} - v_{ij\,IJ} u^{ij}{}_{KL} = 0. \tag{2.30}$$

In addition, the authors in Refs. [37, 41] also derived the following convenient properties,

$$\left(u^{ij}{}_{IM} u_{KL}{}^{JM} - v^{ij\,IM} v_{KL\,JM}\right)\Big|_{[IJ]} = \frac{2}{3}\delta^{[i}{}_{[k} \left(u^{j]m}{}_{IM} u_{l]m}{}^{JM} - v^{j]m\,IM} v_{l]m\,JM}\right)\Big|_{[IJ]} \tag{2.31}$$

$$\begin{aligned}\left(u^{ij}{}_{IJ} v_{KL\,KL} - v^{ij\,IJ} u_{KL}{}^{KL}\right)\Big|_{[IJKL]^+} &= \frac{2}{3}\delta^{[i}{}_{[k} \left(u^{j]m}{}_{IJ} v_{l]m\,KL} - v^{j]m\,IJ} u_{l]m}{}^{KL}\right)\Big|_{[IJKL]^+} \\ &\quad - \frac{1}{12}\delta^{ij}_{KL} \left(u^{mn}{}_{IJ} v_{mn\,KL} - v^{mn\,IJ} u_{mn}{}^{KL}\right)\Big|_{[IJKL]^+},\end{aligned} \tag{2.32}$$

where $|_{[IJ]}$ denotes antisymmetrized indices $[IJ]$ and $|_{[IJKL]^+}$ represents the projection onto the selfdual part.

Secondly, one obtains some properties for the **56** vector $\hat{\mathcal{V}}^{\mathcal{M}}{}_{ij}$. Its $E_{7(7)}$ index is raised and lowered with the symplectic form $\Omega_{\mathcal{M}\mathcal{N}}$, whose components are also given in the SL(8,$\mathbb{R}$) decomposition (Eq. (2.5)),

$$\Omega_{\mathcal{M}\mathcal{N}} = \left(\Omega_{IJ\,KL},\ \ \Omega_{IJ}{}^{KL},\ \ \Omega^{IJ}{}_{KL},\ \ \Omega^{IJ\,KL}\right) = \left(0,\ \ -\delta^{KL}_{IJ},\ \ \delta^{IJ}_{KL},\ \ 0\right).$$

The inverse symplectic form is then easily obtained by requiring $\Omega^{\mathcal{M}\mathcal{P}}\Omega_{\mathcal{P}\mathcal{N}} = \delta^{\mathcal{M}}_{\mathcal{N}}$:

$$\Omega^{\mathcal{M}\mathcal{N}} = \left(\Omega^{IJ\,KL},\ \ \Omega^{IJ}{}_{KL},\ \ \Omega_{IJ}{}^{KL},\ \ \Omega_{IJ\,KL}\right) = \left(0,\ \ \delta^{IJ}_{KL},\ \ -\delta^{KL}_{IJ},\ \ 0\right).$$

Let us now lower the 56-dimensional index of the 56-bein,[5]

$$\hat{\mathcal{V}}_{\mathcal{M}\,ij} = \Omega_{\mathcal{M}\mathcal{N}}\hat{\mathcal{V}}^{\mathcal{N}}{}_{ij} = \left(-\hat{\mathcal{V}}_{IJ\,ij}, \quad \hat{\mathcal{V}}^{IJ}{}_{ij}\right). \tag{2.33}$$

One finally shows that $\hat{\mathcal{V}}$ indeed fulfills the usual vielbein equations [54],

$$\hat{\mathcal{V}}_{\mathcal{M}}{}^{ij}\hat{\mathcal{V}}^{\mathcal{M}}{}_{kl} = i\delta^{ij}_{kl}, \qquad \hat{\mathcal{V}}_{\mathcal{M}\,ij}\hat{\mathcal{V}}^{\mathcal{M}}{}_{kl} = 0, \qquad \hat{\mathcal{V}}_{\mathcal{N}}{}^{ij}\hat{\mathcal{V}}^{\mathcal{M}}{}_{ij} - \hat{\mathcal{V}}^{\mathcal{M}\,ij}\hat{\mathcal{V}}_{\mathcal{N}\,ij} = i\delta^{\mathcal{M}}_{\mathcal{N}}. \tag{2.34}$$

These properties reflect the terminology of calling $\hat{\mathcal{V}}$ a 56-bein of $E_{7(7)}$.

[5]The SU(8) indices can again be raised via complex conjugation in the same manner as in Eq. (2.10).

3. 11-Dimensional Supergravity

This chapter presents 11-dimensional supergravity — the highest dimensional supergravity theory, which was first introduced in 1978 [33, 34]. The corresponding field content of the Lagrangian is a graviton described by the 11-dimensional metric, a Majorana spin 3/2 field and a three-form potential. The total number of bosonic and fermionic degrees of freedom equal the respective numbers in $N = 8$ supergravity. This is crucial for the compactification of 11-dimensional supergravity and the corresponding uplift, which are described in the following Chapters.

The explicit identification of the 11-dimensional fields with the four-dimensional ones within the framework of Kaluza-Klein theory requires a careful comparison of the respective supersymmetry transformations in both theories. This can only be achieved with a *non-linear* redefinition of the 11-dimensional fields. Therefore, the general guideline is SU(8) and $\mathrm{E}_{7(7)}$ covariance: As is the case in $N = 8$ supergravity, the supersymmetry transformations of all redefined fields must be manifestly SU(8) covariant. Furthermore, one must bring the vector and scalar degrees of freedom each into a fundamental **56** representation of $\mathrm{E}_{7(7)}$ in order to relate them with the scalar 56-bein $\mathcal{V}(x)$ and the vectors ${A_\mu}^{\mathcal{M}}(x)$ of $N = 8$ supergravity.

The first part describes the Lagrangian of 11-dimensional supergravity and the associated equations of motion. Section 3.2 introduces the dual fields that are required for the non-linear SU(8) and $\mathrm{E}_{7(7)}$ reformulations of the bosonic fields in Sections 3.3 and 3.4. Finally, Section 3.5 presents the supersymmetry transformations of the redefined fields, which then look quite similar to the corresponding transformations in $N = 8$ supergravity.

3.1. The Lagrangian and the Equations of Motion

The 11-dimensional spacetime is described by a metric $g_{MN}(z)$ for a given coordinate chart z^M. Using the Cartan formalism, the metric gives rise to an elfbein ${E_M}^A(z)$,

$$g_{MN} = \delta_{AB} {E_M}^A {E_N}^B, \tag{3.1}$$

where δ_{AB} is the flat euclidean metric of the 11-dimensional tangent space (with tangent space indices $A, B, \ldots$ and imaginary time). Furthermore, the metric defines the Riemann and Ricci curvature tensors ${R^M}_{NPQ}(z)$, $R_{MN}(z) = {R^P}_{MPN}(z)$ as well as the

Ricci scalar $R(z) = g^{MN}(z)R_{MN}(z)$ in the usual way. Finally, the volume form of the 11-dimensional spacetime is

$$\mathrm{d}V = \sqrt{|g|}\,\mathrm{d}^{11}z,$$

where $g = \det(g_{MN})$ denotes the determinant of the metric. The latter is related to the determinant of the elfbein $E = \det\left(E_M{}^A\right)$ via $|g| = E^2$. With these definitions, the Einstein-Hilbert action for the spacetime reads

$$S = -\frac{1}{2}\int ER\,\mathrm{d}^{11}z. \tag{3.2}$$

The normalization is consistent with the general action in Eq. (3.7).

The fermions in 11-dimensional supergravity are described by a Majorana spin 3/2 field. This can be either described by a spinor $\Psi_A(z)$ on the tangent space or by a spinor $\Psi_M(z) = E_M{}^A(z)\Psi_A(z)$ on the curved manifold. Both spinors contain the same fermionic degrees of freedom and are simultaneously used below. For fixed indices A, M, these fields have 32 components. The corresponding 32×32 matrices $\tilde{\Gamma}_M(z) = E_M{}^A(z)\tilde{\Gamma}_A$ satisfy the Clifford algebra (either on the tangent space or on the curved spacetime),

$$\left\{\tilde{\Gamma}_A, \tilde{\Gamma}_B\right\} = 2\delta_{AB}\,\mathbb{I}_{32\times 32}, \qquad \left\{\tilde{\Gamma}_M, \tilde{\Gamma}_N\right\} = 2g_{MN}\,\mathbb{I}_{32\times 32}. \tag{3.3}$$

Note that tangent space indices $A, B, \ldots$, are raised and lowered with the flat metric δ_{AB} and the curved indices M, N with the spacetime metric $g_{MN}(z)$. It is also convenient to define the 32×32 matrices

$$\tilde{\Gamma}_{A_1\cdots A_i} = \tilde{\Gamma}_{[A_1}\ldots\tilde{\Gamma}_{A_i]}, \qquad \tilde{\Gamma}_{M_1\cdots M_i} = \tilde{\Gamma}_{[M_1}\ldots\tilde{\Gamma}_{M_i]}. \tag{3.4}$$

In the following, antisymmetrized brackets are defined such that e.g.

$$\tilde{\Gamma}_{[ABC]} = \frac{1}{3!}\left(\tilde{\Gamma}_{ABC} + \tilde{\Gamma}_{BCA} + \tilde{\Gamma}_{CAB} - \tilde{\Gamma}_{ACB} - \tilde{\Gamma}_{BAC} - \tilde{\Gamma}_{CBA}\right).$$

Using these antisymmetrized products of Γ matrices, an important fermionic four-form is $X_{(4)}(z)$ with components[1]

$$X^{MNPQ} = 4\sqrt{2}\left(\bar{\Psi}_R\tilde{\Gamma}^{MNPQRS}\Psi_S + 12\bar{\Psi}^{[M}\tilde{\Gamma}^{NP}\Psi^{Q]}\right), \tag{3.5}$$

which is defined on the curved manifold. Here, $\bar{\Psi}_M = \Psi_M^\dagger\tilde{\Gamma}_0$ denotes the respective adjoint spinor.

Finally, 11-dimensional supergravity contains a fully antisymmetric three-form potential $A_{(3)}(z)$ with components A_{MNP}. Such a potential defines a four-form field-strength

$$F_{(4)}(z) = \mathrm{d}A_{(3)}(z) \qquad \Leftrightarrow \qquad F_{MNPQ} = 4!\,\partial_{[M}A_{NPQ]} \tag{3.6}$$

[1]This definition differs from the one in Ref. [46] by a factor of $4\sqrt{2}$.

in the usual way [38]. The partial derivative is with respect to the z-coordinates, $\partial_M = \partial/\partial z^M$.

The Lagrangian of 11-dimensional supergravity is given in terms of the elfbein, the Majorana spin-3/2 fermion and the three-form potential. Using the conventions of Ref. [55], the corresponding action reads

$$\begin{aligned} S = \int \mathrm{d}^{11}z \Big[-\frac{1}{2}ER - \frac{1}{2}E\bar{\Psi}_M\tilde{\Gamma}^{MNP}D_N\Psi_P - \frac{1}{48}EF_{MNPQ}F^{MNPQ} \\ - \frac{i}{12^3\sqrt{2}}\epsilon^{M_1\cdots M_{11}}F_{M_1\cdots M_4}F_{M_5\cdots M_8}A_{M_9\cdots M_{11}} - \frac{1}{24}EF_{MNPQ}X^{MNPQ}\Big]. \end{aligned} \tag{3.7}$$

The first term denotes the Einstein-Hilbert action. The second and third terms represent the kinetic energy for the fermions and the three-form respectively[2]. The fourth and fifth terms represent the interactions — one for the gauge field itself ($\epsilon^{M_1\cdots M_{11}}$ is a tensor) and one for the interaction between fermions and the gauge field.

The 11-dimensional field equations are the corresponding Euler-Lagrange equations of the action in Eq. (3.7). First of all, the variation with respect to the 11-dimensional metric leads to the Einstein equations, which relates the Ricci curvature tensor to the field-strength,

$$R_{MN} = \frac{1}{72}g_{MN}F_{PQRS}F^{PQRS} - \frac{1}{6}F_{MPQR}F_N^{PQR} + \text{ fermionic terms.} \tag{3.8}$$

Note that this thesis is devoted to the bosonic uplift of $N = 8$ supergravity. Hence, fermionic terms are not taken into account. Secondly, consider the variation of the action in Eq. (3.7) with respect to the three-form potential. The corresponding equations of motion are the Maxwell equations in 11 dimensions,

$$D_M\left(F^{MNPQ} + X^{MNPQ}\right) = \frac{\sqrt{2}i}{1152}\,\epsilon^{NPQR_1\ldots R_8}F_{R_1\ldots R_4}F_{R_5\ldots R_8}. \tag{3.9}$$

The fermionic term is retained in this case, as it is crucial for the definition of the dual six-form potential in the next section. Of course, when testing the bosonic uplift relations in Chapters 5, 6 and 7, the fermionic terms will be neglected in the same way as was done to derive the bosonic Einstein equations in Eq. (3.8).

3.2. Dual Fields

This short section introduces a ‘dual six-form potential’ that is essential for the SU(8) reformulation of the 11-dimensional fields in the next section. Such a six-form is obtained

[2] Here, D_M is the covariant derivative. The definition of the action on the spinor Ψ_M is given in Ref. [36].

from the Maxwell equations above and its definition will also contain the fermionic four-form $X_{(4)}$. Without fermions here, the supersymmetry transformations of some redefined fields in the next section would not be SU(8) covariant. Thus, in the meantime it is crucial to maintain the fermionic terms in the Maxwell equations.

Let us dualize Eq. (3.9):

$$8! D_{[M_1} \left[\frac{i}{4!} \epsilon_{M_2 \cdots M_8]NPQR} \left(F^{NPQR} + X^{NPQR} \right) \right] = \frac{7!}{\sqrt{2}} \left(\frac{8!}{4! \cdot 4!} \right) F_{[M_1 \cdots M_4} F_{M_5 \cdots M_8]}. \quad (3.10)$$

This equation can be written in terms of differential forms. First, the rhs includes the usual definition of the wedge product and the lhs includes the usual definition of the exterior derivative, which was already used in Eq. (3.6). Indeed, one may replace the covariant derivative by the partial derivative because of the antisymmetrization. Finally, one defines the Hodge duals for the four-forms on the lhs of Eq. (3.10):[3]

$$F_{(7)}(z) = \star F_{(4)}(z) \qquad \Leftrightarrow \qquad F_{M_1 \cdots M_7} = \frac{i}{4!} \epsilon_{M_1 \cdots M_{11}} F^{M_8 \cdots M_{11}} \quad (3.11)$$

with $\star$ denoting the Hodge star operator. Similarly, one defines the Hodge dual for the fermionic four-form. Hence, Eq. (3.10) simplifies to

$$\mathrm{d} \left(F_{(7)} + \star X_{(4)} \right) = \frac{7!}{\sqrt{2}} F_{(4)} \wedge F_{(4)}.$$

Finally, using Eq. (3.6), the above equation reduces further to

$$\mathrm{d} \left(F_{(7)} + \star X_{(4)} - 3\sqrt{2} A_{(3)} \wedge F_{(4)} \right) = 0.$$

In other words, there locally exists a six-form potential $A_{(6)}(z)$ with components $A_{M_1 \cdots M_6}$, which defines this seven-form,

$$F_{(7)} = \mathrm{d}A_{(6)} + 3\sqrt{2} A_{(3)} \wedge F_{(4)} - \star X. \quad (3.12)$$

The six-form is said to be the dual to the three-form potential and later, it is essential for the SU(8) reformulation of the bosonic fields. Indeed, some of the scalar and vector degrees of freedom are better described by components of the six-form rather than by components of the three-form potential. As a final remark, the Lagrangian could also be written in terms of the dual fields $A_{(6)}(z)$ and $F_{(7)}(z)$ instead of $A_{(3)}(z)$ and $F_{(4)}(z)$ [56].

[3] The i factor in this definition occurs due to the imaginary time convention.

3.3. Non-Linear SU(8) Reformulation of the 11-Dimensional Fields

This section deals with the non-linear reformulation of the fields presented above — the elfbein $E_M{}^A$, the Majorana fermion Ψ_M and the three-form potential A_{MNP} — such that the resulting supersymmetry transformations can be compared to those of $N = 8$ supergravity. For the fermions, this can be achieved by requiring SU(8) covariance. For the scalar and vector fields, this is not sufficient. One must also combine them each into a fundamental **56** representation of $\mathrm{E}_{7(7)}$. In particular, this requires to describe certain bosonic degrees of freedom by some components of the dual six-form potential $A_{M_1\cdots M_6}$. The non-linear field redefinition splits into two parts: This section identifies the various components of the elfbein and the form potentials with the actual scalar and vector degrees of freedom of the theory. The following section then explains, how these constitute fundamental **56** representations of $\mathrm{E}_{7(7)}$.

The SU(8) and $\mathrm{E}_{7(7)}$ reformulation is based on a $4 + 7$ split. Indeed, this is most convenient for the compactification of an 11-dimensional spacetime to four dimensions,

$$\mathcal{M}_{11} \to \mathcal{M}_4 \times \mathcal{M}_7.$$

The set of coordinates z^M splits into four spacetime (external) coordinates x^μ and seven internal coordinates y^m (or in the tangent space: $z^A = (x^\alpha, y^a)$). In the following, capital Roman letters denote 11-dimensional indices. These split into external (Greek letters) and internal indices (lower case Roman letters). Again, letters from the middle of an alphabet always denote curved spacetime indices and letters from the beginning are the corresponding tangent space indices.

Let us describe how the fields behave under this formal coordinate split. The elfbein takes the form

$$E_M{}^A = \begin{pmatrix} e_\mu{}^\alpha & B_\mu{}^m\, e_m{}^a \\ 0 & e_m{}^a \end{pmatrix} \tag{3.13}$$

by partially breaking the local SO(1,10) Lorentz invariance to SO(1,3)×SO(7). Here, $e_\mu{}^\alpha(x, y)$ is proportional to the vierbein of the four-dimensional spacetime and $e_m{}^a(x, y)$ represents the siebenbein of the internal compact space. The latter defines the internal metric,

$$g_{mn} = \delta_{ab} e_m{}^a e_n{}^b, \tag{3.14}$$

where δ_{ab} is the flat euclidean metric on the internal tangent space. This metric (or siebenbein $e_m{}^a$) describe 28 scalar degrees of freedom. In general, the upper off-diagonal of the elfbein can not be gauged to zero. The seven vector fields are denoted as $B_\mu{}^m(x, y)$.

The first field redefinition concerns the vierbein $e_\mu{}^\alpha(x, y)$ as its supersymmetry transformation is not manifestly SU(8) covariant [55]. Let us perform a Weyl rescaling according to

$$e_\mu{}^\alpha = \Delta^{-1/2} \mathring{e}_\mu{}^\alpha, \tag{3.15}$$

where $\Delta(x,y)$ is called the warp factor. It is defined as

$$\Delta = \sqrt{\frac{\det(g_{mn})}{\det(\mathring{g}_{mn})}} = \frac{\det({e_m}^a)}{\det({\mathring{e}_m}^a)}, \tag{3.16}$$

where ${\mathring{e}_m}^a(x,y)$ is any orthonormal frame on the internal space with corresponding metric $\mathring{g}_{mn}(x,y)$. This Weyl rescaling has two convenient properties: On the one hand, the supersymmetry transformation for ${\mathring{e}_\mu}^\alpha(x,y)$ is manifestly SU(8) covariant, see Eq. (3.41). On the other hand, plugging the elfbein ${E_M}^A(x,y)$ into the action in Eq. (3.7) yields a term that corresponds to the *exact* four-dimensional Einstein-Hilbert action in terms of the spacetime $\mathring{g}_{\mu\nu}(x,y)$. Not accidently, the notation for the vierbein and the spacetime metric coincides with the one in $N=8$ supergravity — within the compactification of 11-dimensional supergravity in Chapter 4, these fields will be identified via

$$ {\mathring{e}_\mu}^\alpha(x,y) = {\mathring{e}_\mu}^\alpha(x), \qquad \mathring{g}_{\mu\nu}(x,y) = \mathring{g}_{\mu\nu}(x). $$

Now however, one must still view ${\mathring{e}_\mu}^\alpha(x,y)$ as part of the elfbein above.

Let us now discuss the reformulation of the fermionic fields, before redefining the seven vector fields ${B_\mu}^m$ and the internal siebenbein ${e_m}^a$. The Majorana field decomposes under the $4+7$ split as

$$ \Psi_A \to \begin{pmatrix} \Psi_\alpha, & \Psi_a \end{pmatrix}, \qquad \Psi_M \to \begin{pmatrix} \Psi_\mu, & \Psi_m \end{pmatrix}, $$

where the suppressed spinorial indices run from 1 to 32. It is now more convenient to replace each such spinor index by a pair of indices $(\hat{\alpha},\hat{A})$, where $\hat{\alpha}=1,\ldots 4$ and $\hat{A}=1,\ldots 8$. Most often, these pairs of spinor indices are also suppressed,

$$ \Psi_\mu = (\Psi_\mu)_{\hat{\alpha},\hat{A}}, \qquad \Psi_m = (\Psi_m)_{\hat{\alpha},\hat{A}}. $$

The next two steps are in order to bring these fermionic degrees of freedom into the irreducible representations **8** and **56** of a chiral SU(8) group. Within the compactification in Chapter 4, these will then be identified with the corresponding chiral fermions ${\phi_\mu}^i(x)$ and $\chi^{ijk}(x)$ of $N=8$ supergravity.

In a first step, one defines the chiral SU(8) group and its generators. The 11-dimensional Γ matrices can be written as

$$ \tilde{\Gamma}_A \to \begin{pmatrix} \tilde{\Gamma}_\alpha = \gamma_\alpha \otimes \mathbb{I}_{8\times 8}, & \tilde{\Gamma}_a = \gamma_5 \otimes \Gamma_a \end{pmatrix} $$

using the above index-split [36]. Here, the lower case γ matrices are the 4×4 matrices already defined in Section 2.1. They carry suppressed indices $\hat{\alpha},\hat{\beta}$. Furthermore, the capital Γ matrices denote the seven-dimensional 8×8 matrices with suppressed matrix indices $\hat{A}$, $\hat{B}$, hence $\Gamma_a = (\Gamma_a)_{\hat{A}\hat{B}}$. These flat matrices belong to the internal tangent space and they define curved Γ matrices $\Gamma_m = e_m{}^a\Gamma_a$ using the internal siebenbein. Both kinds of Γ matrices fulfill the corresponding Clifford algebra

$$\{\Gamma_a, \Gamma_b\} = 2\delta_{ab}\mathbb{I}_{8\times 8}, \qquad \{\Gamma_m, \Gamma_n\} = 2g_{mn}\mathbb{I}_{8\times 8}. \tag{3.17}$$

Now, one enlarges the SO(7) tangent space for the Majorana fermions to SU(8). Therefore, the antisymmetrized products of Γ matrices are

$$\Gamma_{a_1\dots a_i} = \Gamma_{[a_1}\dots\Gamma_{a_i]}, \qquad \Gamma_{m_1\dots m_i} = \Gamma_{[m_1}\dots\Gamma_{m_i]}. \tag{3.18}$$

Some useful identities for these traceless 8×8 matrices are given in Appendix A. Most importantly,

$$\Gamma_a, \qquad \Gamma_{ab}, \qquad \Gamma_{abc}$$

are 63 independent matrices and hence, generate a chiral SU(8) group acting on the Majorana spinors above.

In a second step, one rewrites the fermionic Majorana spinors into irreducible SU(8) representations. Therefore, a Weyl rescaling of $\Psi_\alpha(x,y)$ and $\Psi_a(x,y)$ results in [55]

$$\phi'_\mu = \mathring{e}_\mu{}^\alpha\Delta^{-1/4}(i\gamma_5)^{-1/2}\left(\Psi_\alpha - \frac{1}{2}\gamma_5\gamma_\alpha\Gamma^a\Psi_a\right), \qquad \phi'_m = e_m{}^a\Delta^{-1/4}(i\gamma_5)^{-1/2}\Psi_a. \tag{3.19}$$

This may need some explanation: All spinors in the above equations (ϕ'_μ, ϕ'_a, Ψ_α, Ψ_a) have 32 spinor components, labeled each by an index pair $(\hat\alpha, \hat A)$ as explained above. However, these indices are suppressed, e.g.

$$\phi'_\mu = \left(\phi'_\mu\right)_{\hat\alpha,\hat A}.$$

Then, the four-dimensional γ matrices act on the $\hat\alpha$ index of these spinors and the 8×8 Γ matrices act on the $\hat A$ index. For example, the first of the above equations actually reads

$$\left(\phi'_\mu\right)_{\hat\alpha,\hat A} = \mathring{e}_\mu{}^\alpha\Delta^{-1/4}\left[(i\gamma_5)^{-1/2}\right]_{\hat\alpha\hat\beta}\left((\Psi_\alpha)_{\hat\beta,\hat A} - \frac{1}{2}(\gamma_5)_{\hat\beta\hat\gamma}(\gamma_\alpha)_{\hat\gamma\hat\delta}(\Gamma^a)_{\hat A\hat B}(\Psi_a)_{\hat\delta,\hat B}\right).$$

One already notes that it is rather confusing to carry all these indices. So if possible, they are suppressed in the following. The above primed spinors represent only an intermediate step. Another projection onto their chiral components finally gives

$$\begin{aligned}(\phi_\mu)_{\hat\alpha}{}^{\hat A} &= \frac{1}{2}(1+\gamma_5)_{\hat\alpha\hat\beta}\left(\phi'_\mu\right)_{\hat\beta,\hat A},\\ (\phi_\mu)_{\hat\alpha\,\hat A} &= \frac{1}{2}(1-\gamma_5)_{\hat\alpha\hat\beta}\left(\phi'_\mu\right)_{\hat\beta,\hat A},\end{aligned} \tag{3.20}$$

$$\begin{aligned}\chi_{\hat\alpha}{}^{\hat A\hat B\hat C} &= \frac{3}{4}\sqrt{2}i(1+\gamma_5)_{\hat\alpha\hat\beta}\Gamma^m{}_{[\hat A\hat B}\left(\phi'_m\right)_{\hat\beta,\hat C]},\\ \chi_{\hat\alpha\,\hat A\hat B\hat C} &= \frac{3}{4}\sqrt{2}i(1-\gamma_5)_{\hat\alpha\hat\beta}\Gamma^m{}_{[\hat A\hat B}\left(\phi'_m\right)_{\hat\beta,\hat C]}.\end{aligned} \tag{3.21}$$

Hence, the spinor $(\phi_\mu)_{\hat\alpha}{}^{\hat A}(x,y)$ belongs to the **8** and the trispinor $\chi_{\hat\alpha}{}^{\hat A\hat B\hat C}(x,y)$ to the **56** representation of the chiral SU(8) group[4]. Note that these are highly non-linear in the 11-dimensional fields — they will be related to the corresponding fermions of $N = 8$ supergravity in Chapter 4.

Most expressions in this thesis do not carry the four-dimensional indices $\hat\alpha, \hat\beta$. However, the eight-dimensional indices $\hat A, \hat B$ that label the chiral SU(8) group are continuously used within the whole work. It is therefore more convenient to replace them by unhatted indices

$$\hat A, \hat B \to A, B.$$

Of course, this should not cause confusion with the 11-dimensional tangent space indices. However, it will always be clear from the context whether $A, B, \ldots$ are SU(8)- or 11-dimensional tangent space indices. Hence, within the remainder of this thesis,

$$(\phi_\mu)_{\hat\alpha}{}^{\hat A} = \phi_\mu{}^A, \qquad \chi_{\hat\alpha}{}^{\hat A\hat B\hat C} = \chi^{ABC}.$$

The last part of this section is devoted to the non-linear SU(8) reformulation for the bosonic degrees of freedom that are hidden in the elfbein and the form-potentials. Let us first discuss the vector fields $B_\mu{}^m(x,y)$ and the siebenbein $e_m{}^a(x,y)$. The vectors already transform in a manifestly SU(8) covariant way. However, for later convenience, they are rescaled according to

$$\mathcal{B}_\mu{}^m = -\frac{1}{2}B_\mu{}^m. \tag{3.22}$$

The scalar degrees of freedom of the internal siebenbein can be reformulated as well,

$$\mathcal{V}^m{}_{AB} = -\frac{\sqrt{2}}{8}\Delta^{-1/2}e^m{}_a\Gamma^a{}_{AB} = -\frac{\sqrt{2}}{8}\Delta^{-1/2}\Gamma^m{}_{AB}. \tag{3.23}$$

This reformulation was found by requiring SU(8) covariance of the corresponding supersymmetry transformations. In the following and in the next section, these vector and scalar fields both are extended to **56** vectors $\mathcal{B}_\mu{}^{\mathcal{M}}$ and $\mathcal{V}^{\mathcal{M}}{}_{AB}$ of an $\mathrm{E}_{7(7)}$ symmetry[5]. This is the key to relate them to the corresponding scalars and vectors of $N = 8$ supergravity (which also form **56** representations of the $\mathrm{E}_{7(7)}$ group of duality invariance).

The remaining components of $\mathcal{B}_\mu{}^{\mathcal{M}}(x,y)$ and $\mathcal{V}^{\mathcal{M}}{}_{AB}(x,y)$ are non-linear combinations of the various components of the three-form potential under the $4+7$ split,

$$A_{MNP} = \Big(A_{\mu\nu\rho}, \quad A_{\mu\nu m}, \quad A_{\mu mn}, \quad A_{mnp}\Big)\,.$$

First, the components $A_{\mu mn}$ represent 21 vector fields and do not contain any scalar degrees of freedom. Since its supersymmetry transformation is not manifestly SU(8)

[4]The chiral SU(8) indices $\hat A, \hat B, \ldots$ are raised and lowered by complex conjugation.

[5]The normalizations in Eqs. (3.22, 3.23) as well as in the following redefinitions have been chosen accordingly.

covariant, it is more convenient to define the vector fields

$$\mathcal{B}_{\mu\, mn} = -3\sqrt{2}\left(A_{\mu\, mn} - B_\mu{}^p A_{mnp}\right). \tag{3.24}$$

They are a linear combination of the vectors $B_\mu{}^m$ and $A_{\mu mn}$, but *non-linear* in all 11-dimensional fields.

Secondly, the components $A_{\mu\nu\rho}$ and $A_{\mu\nu m}$ describe both, scalar and vector degrees of freedom, all of which are also contained in the components of the dual six-form potential, $A_{m_1\cdots m_6}$ and $A_{\mu m_1\cdots m_5}$. The former represent seven scalar fields and the latter 21 vector degrees of freedom. Let us now discuss the vector fields. In order to make the corresponding supersymmetry transformations manifestly SU(8) covariant, one defines

$$\mathcal{B}_\mu{}^{mn} = -3\sqrt{2}\mathring{\eta}^{mnp_1\cdots p_5}\left(A_{\mu\, p_1\cdots p_5} - B_\mu{}^q A_{q\, p_1\cdots p_5} - \frac{\sqrt{2}}{4}\left(A_{\mu\, p_1 p_2} - B_\mu{}^q A_{qp_1p_2}\right) A_{p_3p_4p_5}\right). \tag{3.25}$$

Here, $\mathring{\eta}^{m_1\cdots m_7}$ is the seven-dimensional Levi-Cevita tensor density (hence, a number $\pm 1, 0$). Again, $\mathcal{B}_\mu{}^{mn}$ combines several fields including the vectors $A_{\mu m_1\cdots m_5}$.

Finally, the remaining bosonic degrees of freedom of the three-form potential are described by A_{mnp} and $A_{m_1\cdots m_6}$. A_{mnp} represents 35 and $A_{m_1\cdots m_6}$ seven scalar fields. Again, the corresponding supersymmetry transformations can be made SU(8) covariant by redefining

$$\mathcal{V}_{mn\, AB} = \frac{\sqrt{2}}{8}\Delta^{-1/2}\left(\Gamma_{mn\, AB} + 6\sqrt{2}A_{mnp}\Gamma^p_{AB}\right), \tag{3.26}$$

$$\begin{aligned}\mathcal{V}^{mn}{}_{AB} = -\frac{\sqrt{2}}{8}\cdot\frac{1}{5!}\mathring{\eta}^{mnp_1\cdots p_5}\Delta^{-1/2}\Big[&\Gamma_{p_1\cdots p_5\, AB} + 60\sqrt{2}A_{p_1p_2p_3}\Gamma_{p_4p_5\, AB} \\ &- 6!\sqrt{2}\left(A_{qp_1\cdots p_5} - \frac{\sqrt{2}}{4}A_{qp_1p_2}A_{p_3p_4p_5}\right)\Gamma^q_{AB}\Big]\end{aligned} \tag{3.27}$$

in a non-linear way. Note that $\mathcal{V}_{mn\, AB}$ differs from the respective component in Ref. [54] by a factor of -1. This is explained in the following section.

3.4. E_7 Structures in the Bosonic 11-Dimensional Fields

This section shows how the vector and scalar fields presented above can be extended to form $\mathrm{E}_{7(7)}$ covariant objects. One first counts the number of vector degrees of freedom: $7 + 21$ vector fields $\mathcal{B}_\mu{}^m$ and $\mathcal{B}_{\mu\, mn}$ as well as 21 dual vectors $\mathcal{B}_\mu{}^{mn}$. This suggests to combine all of these into a fundamental **56** representation of $\mathrm{E}_{7(7)}$ since its SL(8,ℝ) and GL(7,ℝ) decompositions are given by

$$\mathbf{56} \to \mathbf{28} \oplus \overline{\mathbf{28}} \to \mathbf{7} \oplus \mathbf{21} \oplus \overline{\mathbf{21}} \oplus \overline{\mathbf{7}}. \tag{3.28}$$

Quite naturally, one defines the $E_{7(7)}$ vector $\mathcal{B}_\mu{}^{\mathcal{M}}(x,y)$ with 56 components, such that under the above decomposition,

$$\mathcal{B}_\mu{}^{\mathcal{M}} = \left(\mathcal{B}_\mu{}^{\mathtt{MN}},\quad \mathcal{B}_{\mu\,\mathtt{MN}}\right) = \left(\mathcal{B}_\mu{}^{m},\quad \mathcal{B}_\mu{}^{mn},\quad \mathcal{B}_{\mu\,mn},\quad \mathcal{B}_{\mu\,m}\right).$$

More explicitly, the antisymmetric SL(8,$\mathbb{R}$) indices $\mathtt{M},\mathtt{N}$ run from 1 to 8 and components for $\mathtt{M}=8$ are abbreviated by

$$\mathcal{B}_\mu{}^{m8} = -\mathcal{B}_\mu{}^{8m} = \mathcal{B}_\mu{}^{m}, \qquad \mathcal{B}_{\mu\,m8} = -\mathcal{B}_{\mu\,8m} = \mathcal{B}_{\mu\,m}.$$

However, there is a serious problem here. There are no more vector degrees of freedom in 11-dimensional supergravity to link with the remaining seven vector fields $\mathcal{B}_{\mu\,m}(x,y)$. The usual solution is taking the dualization of gravity into account.

Dualizing gravity is formally possible at the linearized level[6]. Therefore, one approximates the 11-dimensional metric,

$$g_{MN} = \eta_{MN} + h_{MN} + \mathcal{O}(h^2). \tag{3.29}$$

The dual field to $h_{MN}(x,y)$ is called 'dual graviton' — it is represented by a field $A_{M_1\cdots M_8|N}(x,y)$, which belongs to the (8,1) representation of $\mathrm{GL}(11,\mathbb{R})$[7]. This field decomposes into the various components under the $4+7$ split and indeed, there is one component that contains seven *artificial* vector degrees of freedom: $A_{\mu\,m_1\cdots m_7|n}$. Its supersymmetry transformations are not SU(8) covariant but a non-linear combination with the various other vector and scalar fields transforms in a manifestly SU(8) covariant way. With the correct normalization, this can then be identified with $\mathcal{B}_{\mu\,m}$,

$$\begin{aligned}\mathcal{B}_{\mu\,m} = -18\mathring{\eta}^{p_1\cdots p_7}\Big[&A_{\mu\,p_1\cdots p_7,m} + (3\tilde{C}_0 - 1)\left(A_{\mu\,p_1\cdots p_5} - B_\mu{}^p A_{p\,p_1\cdots p_5}\right)A_{p_6 p_7 m} \\ &+ \tilde{C}_0 A_{p_1\cdots p_6}\left(A_{\mu p_7 m} - B_\mu{}^p A_{p p_7 m}\right) + \frac{\sqrt{2}}{12}\left(A_{\mu p_1 p_2} - B_\mu{}^p A_{p p_1 p_2}\right)A_{p_3 p_4 p_5}A_{p_6 p_7 m}\Big].\end{aligned} \tag{3.30}$$

Here $\tilde{C}_0$ is an undetermined constant.

The scalar fields can be put into an irreducible **56** representation in the same way. As is the case in $N=8$ supergravity, the resulting object is a 56-bein $\mathcal{V}^{\mathcal{M}}{}_{AB}$ that decomposes under the above SL(8,$\mathbb{R}$) and GL(7,$\mathbb{R}$) decompositions as

$$\mathcal{V}^{\mathcal{M}}{}_{AB} = \left(\mathcal{V}^{\mathtt{MN}}{}_{AB},\quad \mathcal{V}_{\mathtt{MN}\,AB}\right) = \left(\mathcal{V}^{m}{}_{AB},\quad \mathcal{V}^{mn}{}_{AB},\quad \mathcal{V}_{mn\,AB},\quad \mathcal{V}_{m\,AB}\right). \tag{3.31}$$

As is the case for the vectors, the $\mathtt{M}=8$ components are abbreviated as

$$\mathcal{V}^{m8}{}_{AB} = -\mathcal{V}^{8m}{}_{AB} = \mathcal{V}^{m}{}_{AB}, \qquad \mathcal{V}_{m8\,AB} = -\mathcal{V}_{8m\,AB} = \mathcal{V}_{m\,AB}.$$

[6]Restricting to the linearized level causes no inconsistency in Chapter 4, since the artificial vector degrees of freedom arising from the dual graviton do not enter the obtained embedding formulae. They are only required to formally find the E_7 structures in the bosonic 11-dimensional fields.

[7]in Ref. [46], $A_{M_1\cdots M_8|N}$ was denoted by $h_{M_1\cdots M_8|N}$.

Again, the last seven components are not yet defined but the counting argument that is used in the vector case above does not apply here. Indeed, all the $70 = 28+7+35$ scalar degrees of freedom in 11-dimensional supergravity (hidden in $e_m{}^a$, $A_{m_1\cdots m_6}$ and A_{mnp}) are already assigned to the first 49 components of $\mathcal{V}^{\mathcal{M}}{}_{AB}$. However, in this case, the supersymmetry transformations of the vector fields $\mathcal{B}_{\mu\, m}$ led to the definition of $\mathcal{V}_{m\,AB}$. In order that Eq. (3.42) holds, $\mathcal{V}_{m\,AB}$ must take the form [54][8]

$$\begin{aligned}\mathcal{V}_{m\;AB} = \frac{\sqrt{2}}{8}\cdot\frac{1}{7!}\mathring{\eta}^{p_1\cdots p_7}\Delta^{-1/2}\Bigg[&(\Gamma_{p_1\cdots p_7}\Gamma_m)_{AB} + 126\sqrt{2}A_{mp_1p_2}\Gamma_{p_3\cdots p_7\;AB}\\ &+3\sqrt{2}\cdot 7!\left(A_{mp_1\cdots p_5}+\frac{\sqrt{2}}{4}A_{mp_1p_2}A_{p_3p_4p_5}\right)\Gamma_{p_6p_7\;AB}\\ &+\frac{9!}{2}\left(A_{mp_1\cdots p_5}+\frac{\sqrt{2}}{12}A_{mp_1p_2}A_{p_3p_4p_5}\right)A_{p_6p_7q}\Gamma^q_{AB}\Bigg].\end{aligned} \tag{3.32}$$

With the above definitions, the supersymmetry transformations for the irreducible **56** representations $\mathcal{B}_\mu{}^{\mathcal{M}}(x,y)$ and $\mathcal{V}^{\mathcal{M}}{}_{AB}(x,y)$ are manifestly SU(8) and $E_{7(7)}$ covariant, see Eq. (3.42, 3.43).

The rest of this section justifies the identification of the **56** vector $\mathcal{V}^{\mathcal{M}}{}_{AB}$ as the 56-bein of $\mathrm{E}_{7(7)}$. The reasoning is similar to the case of the 56-bein $\hat{\mathcal{V}}^{\mathcal{M}}{}_{ij}$ in $N=8$ supergravity. First of all, the **56** index $\mathcal{M}$ is raised and lowered with the symplectic form $\Omega_{\mathcal{M}\mathcal{N}}$, whose components are given in the respective SL(8,$\mathbb{R}$) decomposition (Eq. (2.5)),

$$\Omega_{\mathcal{M}\mathcal{N}} = \left(\Omega_{\mathtt{MN}\,\mathtt{PQ}},\quad \Omega_{\mathtt{MN}}{}^{\mathtt{PQ}},\quad \Omega^{\mathtt{MN}}{}_{\mathtt{PQ}},\quad \Omega^{\mathtt{MN}\,\mathtt{PQ}}\right) = \left(0,\quad -\delta^{\mathtt{PQ}}_{\mathtt{MN}},\quad \delta^{\mathtt{MN}}_{\mathtt{PQ}},\quad 0\right).$$

The inverse symplectic form is then simply

$$\Omega^{\mathcal{M}\mathcal{N}} = \left(\Omega^{\mathtt{MN}\,\mathtt{PQ}},\quad \Omega^{\mathtt{MN}}{}_{\mathtt{PQ}},\quad \Omega_{\mathtt{MN}}{}^{\mathtt{PQ}},\quad \Omega_{\mathtt{MN}\,\mathtt{PQ}}\right) = \left(0,\quad \delta^{\mathtt{MN}}_{\mathtt{PQ}},\quad -\delta^{\mathtt{PQ}}_{\mathtt{MN}},\quad 0\right),$$

such that one has again $\Omega^{\mathcal{M}\mathcal{P}}\Omega_{\mathcal{P}\mathcal{N}} = \delta^{\mathcal{M}}_{\mathcal{N}}$. Lowering the 56-dimensional index of the 56-bein $\mathcal{V}^{\mathcal{M}}{}_{AB}$ yields

$$\mathcal{V}_{\mathcal{M}\,AB} = \Omega_{\mathcal{M}\mathcal{N}}\mathcal{V}^{\mathcal{N}}{}_{AB} = \left(-\mathcal{V}_{\mathtt{MN}\,AB},\quad \mathcal{V}^{\mathtt{MN}}{}_{AB}\right), \tag{3.33}$$

and the chiral SU(8) indices of the 56-bein are again raised via complex conjugation,

$$\mathcal{V}^{\mathcal{M}\,AB} = \left(\mathcal{V}^{\mathcal{M}}{}_{AB}\right)^*, \qquad \mathcal{V}_{\mathcal{M}}{}^{AB} = \left(\mathcal{V}_{\mathcal{M}\,AB}\right)^*. \tag{3.34}$$

[8] As is the case for the definition of $\mathcal{V}_{mn\,AB}$, the above equation for $\mathcal{V}_{m\,AB}$ differs from the respective component in Ref. [54] by a factor of -1. The reason is that in contrast to our definition in Eq. (3.31), the vielbein components in Ref. [54] constitute the **56** vector $\mathcal{V}_{\mathcal{M}\,AB}$ *with lower* $\mathrm{E}_{7(7)}$ *index*. Hence, our definitions are consistent with those of Ref. [54] since the index-raising inserts the corresponding minus sign, (Eq. (3.33)).

One finally shows that the 56-bein $\mathcal{V}^{\mathcal{M}}{}_{AB}$ fulfills the usual vielbein relations that are already satisfied by the four-dimensional 56-bein (Eq. (2.34)),

$$\begin{gathered}\mathcal{V}_{\mathcal{M}}{}^{AB}\mathcal{V}^{\mathcal{M}}{}_{CD} = i\delta^{AB}_{CD}, \qquad\qquad \mathcal{V}_{\mathcal{M}\,AB}\mathcal{V}^{\mathcal{M}}{}_{CD} = 0,\\ \mathcal{V}_{\mathcal{N}}{}^{AB}\mathcal{V}^{\mathcal{M}}{}_{AB} - \mathcal{V}^{\mathcal{M}\,AB}\mathcal{V}_{\mathcal{N}\,AB} = i\delta^{\mathcal{M}}_{\mathcal{N}}.\end{gathered} \tag{3.35}$$

Therefore, one uses some properties of the 8×8 Γ matrices that are presented in Appendix A.

3.5. Supersymmetry Transformations

Let us finally summarize the supersymmetry transformations for the various redefined fields presented above. The detailed derivations were found in Refs. [46, 55], starting from the general 11-dimensional supersymmetry transformations

$$\delta_\epsilon E_M{}^A = \frac{1}{2}\,\bar{\epsilon}\,\tilde{\Gamma}^A\Psi_M, \tag{3.36}$$

$$\delta_\epsilon A_{MNP} = -\frac{\sqrt{2}}{8}\,\bar{\epsilon}\,\tilde{\Gamma}_{[MN}\Psi_{P]}, \tag{3.37}$$

$$\delta_\epsilon \Psi_M = \left[D_M + \frac{\sqrt{2}}{288}\left(\tilde{\Gamma}_{MNPQR} - 8g_{MN}\tilde{\Gamma}_{PQR}\right)F^{NPQR}\right]\epsilon, \tag{3.38}$$

$$\delta_\epsilon A_{M_1\cdots M_6} = -\frac{3}{6!\sqrt{2}}\,\bar{\epsilon}\,\tilde{\Gamma}_{[M_1\cdots M_5}\Psi_{M_6]} + \frac{1}{8}\bar{\epsilon}\,\tilde{\Gamma}_{[M_1 M_2}\Psi_{M_3}A_{M_4 M_5 M_6]}, \tag{3.39}$$

$$\delta_\epsilon A_{M_1\cdots M_8|N} \propto \bar{\epsilon}\tilde{\Gamma}_{M_1\cdots M_8}\Psi_N - \bar{\epsilon}\tilde{\Gamma}_{N[M_1\cdots M_7}\Psi_{M_8]} - C_0\eta_{N[M_1}\bar{\epsilon}\tilde{\Gamma}_{M_2\cdots M_7}\Psi_{M_8]}. \tag{3.40}$$

Here, the supersymmetry transformation parameter is a spinor ϵ with 32 components (also divided into an index pair $(\hat{\alpha}, \hat{A})$). The first three transformations are the well known supersymmetry transformations of 11-dimensional supergravity [33] written in the conventions of Ref. [55]. From these, the corresponding supersymmetry transformation for the dual six-form potential was found using Eq. (3.12) [46]. The transformation for the dual gravity potential $A_{M_1\cdots M_8|N}(x, y)$ can be found in the same way, using the correct dualization procedure for the graviton field. Here, C_0 is some undetermined constant. For the details, the interested reader may consult Ref. [46] and references therein.

One finally obtains the corresponding supersymmetry transformations for the reformulated fields. Therefore, one considers the above relations according to the $4+7$ split and rearranges the fields according to the non-linear reformulations of the previous section. For example, the Weyl rescaled vierbein transforms as

$$\delta_\epsilon \mathring{e}_\mu{}^\alpha = \frac{1}{2}\bar{\epsilon}^A\gamma^\alpha\phi_{\mu\,A} + \text{h.c.}, \tag{3.41}$$

which is manifestly SU(8) covariant [55]. One also notices the similarity to the corresponding supersymmetry transformation of the vierbein in $N = 8$ supergravity. The exact relations between the 11-dimensional and the four-dimensional fields will be given in the next chapter.

In the following, the supersymmetry transformations are only listed for the fully redefined vector and scalar fields $\mathcal{B}_\mu{}^{\mathcal{M}}(x, y)$ and $\mathcal{V}^{\mathcal{M}}{}_{AB}(x, y)$ rather than for the pre-fields according to the $4+7$ split. However, all vector and scalar degrees of freedom constitute these two 56-dimensional representations of $E_{7(7)}$. Hence, the following two equations represent the supersymmetry transformations for *all* the bosonic fields of 11-dimensional supergravity [41, 46, 47]:

$$\delta_\epsilon \mathcal{B}_\mu{}^{\mathcal{M}} = \frac{i}{2} \mathcal{V}^{\mathcal{M}}{}_{AB} X_\mu{}^{AB}(\epsilon) + \text{h.c.} \tag{3.42}$$

$$\delta_\epsilon \mathcal{V}^{\mathcal{M}}{}_{AB} = \sqrt{2} \Sigma_{ABCD}(\epsilon) \mathcal{V}^{\mathcal{M}\,CD}. \tag{3.43}$$

Here, $X_\mu{}^{AB}(\epsilon)$ and the self-dual four-form $\Sigma_{ABCD}(\epsilon)$ of SU(8) are quite similar to the four-dimensional fields in Eqs. (2.18, 2.19),

$$X_\mu{}^{AB}(\epsilon) = 2\sqrt{2}\bar{\epsilon}^A \phi_\mu{}^B + \bar{\epsilon}_C \mathring{e}_\mu{}^\alpha \gamma_\alpha \chi^{ABC}, \tag{3.44}$$

$$\Sigma_{ABCD}(\epsilon) = \bar{\epsilon}_{[A} \chi_{BCD]} + \frac{1}{4!} \epsilon_{ABCDEFGH} \bar{\epsilon}^E \chi^{FGH}. \tag{3.45}$$

In Eq. (3.45), the second term is the Hodge dual of the first term and the ϵ tensor is the corresponding totally anti-symmetric tensor with chiral SU(8) indices. As is the case for the vierbein, the supersymmetry transformations for the redefined 11-dimensional vectors and scalars are quite similar to those of the respective four-dimensional fields. Note that this could only be achieved because of the *non-linear structure* of the redefinitions. The next chapter will explicitly identify these bosonic fields

$$\left(\mathcal{B}_\mu{}^{\mathcal{M}}, \quad \mathcal{V}^{\mathcal{M}}{}_{AB}\right) \Leftrightarrow \left(\mathcal{A}_\mu{}^{\mathcal{M}}, \quad \hat{\mathcal{V}}^{\mathcal{M}}{}_{ij}\right).$$

As in the presentation of $N = 8$ supergravity, the fermionic supersymmetry transformations are not listed here, since they are not required for the bosonic uplift of $N = 8$ supergravity to 11 dimensions. A detailed description can be found in Ref. [55].

4. The Embedding of Gauged $N = 8$ Supergravity into 11 Dimensions

This chapter presents the main part of this thesis: the embedding of gauged $N = 8$ supergravity into 11 dimensions, which is based on the compactification of the extra dimensions on a seven-sphere,

$$\mathcal{M}_{11} \to \mathcal{M}_4 \times S^7. \tag{4.1}$$

The previous chapter found the non-linear field redefinitions $\Phi(x, y) \to F(\Phi(x, y))$, such that the lower and higher-dimensional supersymmetry transformations can now be compared (Eq. (1.4)). This gives a direct correspondence between the reformulated 11-dimensional fields $F(\Phi(x, y))$ and the four-dimensional ones $\Phi^{(n)}(x)$. Based on these relations, the second part of this chapter finally derives the complete *non-linear uplift Ansätze* for the bosonic 11-dimensional fields

$$\Phi(x, y) \Leftrightarrow \left(\Phi^{(n)}(x), \quad Y^{(n)}(y)\right). \tag{4.2}$$

In particular, it shows the dependence of the eigenfunctions $Y^{(n)}(y)$ on the Killing spinors and vectors of the seven-sphere.

First, Section 4.1 gives the known background of 11-dimensional supergravity — the ground state solution $\{\Phi_0(x, y)\}$. It also introduces the required Killing spinors and vectors that belong to the compactified round seven-sphere. Furthermore, Section 4.2 presents the explicit non-linear expansion to a general solution of 11-dimensional supergravity (Eq. (1.4)). Based on this, Section 4.3 derives all bosonic uplift Ansätze in the form of Eq. (4.2).

Finally, Section 4.4 derives two subsequent Ansätze for the internal four-form field-strength $F_{mnpq}(x, y)$ and the Freund-Rubin term $\mathfrak{f}_{\mathrm{FR}}(x, y)$ [48, 49, 50]. The field-strength Ansatz has been found from Eq. (3.6) and the explicit Ansatz for the internal three-form potential. On the other hand, the Freund-Rubin term[1] has been found from the so-called 'generalized vielbein postulate' (GVP) for the 56-bein $\mathcal{V}^{\mathcal{M}}{}_{AB}$ in 11 dimensions. These secondary Ansätze are used in the following chapters in order to explicitly check the consistency of the obtained group invariant 11-dimensional Freund-Rubin solutions.

[1] The Freund-Rubin term is defined as the four-dimensional dual of the external field-strength $F_{\mu\nu\rho\sigma}$, see Eq. (4.58).

4.1. The Ground State Solution of the S^7 Reduction

Let us describe the ground state solution $\{\Phi_0(x, y)\}$ of 11-dimensional supergravity. The background fields are

$$\Phi_0(x, y) = \left(E_M{}^A(x, y)\Big|_{\mathrm{BG}},\quad \Psi_M(x, y)\Big|_{\mathrm{BG}},\quad A_{MNP}(x, y)\Big|_{\mathrm{BG}} \right)$$

and in the following, the different components are discussed: The AdS_4 spacetime is described by the background vierbein $\mathring{e}_\mu{}^\alpha(x)$ and metric $\mathring{g}_{\mu\nu}(x)$. The internal background siebenbein and metric are given by

$$e_m{}^a|_{\mathrm{BG}} = \mathring{e}_m{}^a, \qquad g_{mn}|_{BG} = \mathring{g}_{mn}, \tag{4.3}$$

where $\mathring{e}_m{}^a(y)$ denotes the orthonormal frame for the corresponding S^7 metric $\mathring{g}_{mn}(y)$. The only other field of 11-dimensional supergravity that acquires any background value is $A_{\mu\nu\rho}(x, y)$. In particular, it is related to the internal components $A_{m_1\cdots m_6}(x, y)$ of the dual six-form potential. One finds [47]

$$A_{m_1\cdots m_6}|_{\mathrm{BG}} = -3\sqrt{2}\mathring{\zeta}_{m_1\cdots m_6}, \tag{4.4}$$

where the six-form $\mathring{\zeta}_{m_1\cdots m_6}(y)$ defines the background volume form $\mathring{\eta}^{m_1\cdots m_7}$ of the seven-sphere,

$$7!\mathring{D}_{[m_1}\mathring{\zeta}_{m_2\cdots m_7]} = m_7\, \mathring{\eta}_{m_1\cdots m_7}. \tag{4.5}$$

Here, m_7 is the inverse S^7 radius and $\mathring{D}_m$ denotes the corresponding covariant derivative. All other fields of 11-dimensional supergravity vanish within the background (fermions, vector bosons and the scalars of the internal three-form potential).

The following section presents the expansion of these 11-dimensional fields around the ground state solution above in the sense of Eq. (1.4). In general, the eigenfunctions $Y^{(n)}(y)$ depend on the form of the reduction — in our case, they will depend on the Killing spinors and vectors of the seven-sphere[2], which are now introduced.

The eight Killing spinors $\eta^I(y)$ are chosen to be orthonormal[3],

$$\bar{\eta}^I\eta^J = \delta^{IJ}, \qquad \eta^I\bar{\eta}^I = \mathbb{I}_{8\times 8}. \tag{4.6}$$

They satisfy

$$\left(\mathring{D}_m + \frac{i}{2}m_7\mathring{\Gamma}_m \right)\eta^I = 0, \tag{4.7}$$

[2] Later, the 28 Killing vectors of the SO(8) isometry group of the seven-sphere will be related to the 28 electric vector bosons of maximally gauged $N = 8$ supergravity.

[3] Again, $I, J, \ldots$ are SO(8) indices, whereas the SU(8) indices A, B are suppressed, so $\eta^I = \eta^I_A$. Furthermore, the charge conjugation matrix is set to the identity, which implies that $\bar{\eta}^I = (\eta^I)^\dagger$.

where the $\mathring{\Gamma}$ matrices are the antisymmetric and purely imaginary ($\mathring{\Gamma}^\dagger = \mathring{\Gamma}$) generators of the Clifford algebra in seven dimensions,

$$\left\{\mathring{\Gamma}_m, \mathring{\Gamma}_n\right\} = 2\mathring{g}_{mn}\mathbb{I}_{8\times 8}. \tag{4.8}$$

The Killing spinors also define a set of Killing vectors and their derivatives,

$$K_m{}^{IJ} = i\bar{\eta}^I\mathring{\Gamma}_m\eta^J, \qquad K_{mn}{}^{IJ} = \bar{\eta}^I\mathring{\Gamma}_{mn}\eta^J, \tag{4.9}$$

where the antisymmetrized products of $\mathring{\Gamma}$ matrices are defined as

$$\mathring{\Gamma}_{m_1\ldots m_i} = \mathring{\Gamma}_{[m_1}\ldots\mathring{\Gamma}_{m_i]}. \tag{4.10}$$

In particular, using Eqs. (4.7, 4.8, 4.10), one verifies that $K_{mn}{}^{IJ}$ is indeed, proportional to the derivative of $K_m{}^{IJ}$,

$$\mathring{D}_n K_m{}^{IJ} = m_7 K_{mn}{}^{IJ}, \qquad \mathring{D}_p K_{mn}{}^{IJ} = 2m_7\mathring{g}_{p[m}K_{n]}{}^{IJ}. \tag{4.11}$$

Note that curved seven-dimensional indices of the Killing vectors and their derivatives are always raised and lowered with the background S^7 metric $\mathring{g}_{mn}$.

4.2. Non-Linear Expansion around the Ground State

This section derives the full non-linear expansion of all redefined 11-dimensional fields,

$$F(\Phi(x,y)) = \left(\mathring{e}_\mu{}^\alpha(x,y),\quad \phi_\mu{}^A(x,y),\quad \chi^{ABC}(x,y),\quad B_\mu{}^{\mathcal{M}}(x,y),\quad \mathcal{V}^{\mathcal{M}}{}_{AB}(x,y)\right),$$

around the ground state solution according to Eq. (1.4). This automatically yields the correct relation to the four-dimensional fields

$$\Phi^{(n)}(x) = \left(\mathring{e}_\mu{}^\alpha(x),\quad \phi_\mu{}^i(x),\quad \chi^{ijk}(x),\quad A_\mu{}^{\mathcal{M}}(x),\quad \hat{\mathcal{V}}^{\mathcal{M}}{}_{ij}(x)\right).$$

These relations are then used in the next section to find the explicit uplift formulae for the fields of $N = 8$ supergravity to 11 dimensions.

The simplest example is the relation between the vierbeine of $N = 8$ and 11-dimensional supergravity: [41]

$$\mathring{e}_\mu{}^\alpha(x,y) = \mathring{e}_\mu{}^\alpha(x). \tag{4.12}$$

This relation can be established consistently to all orders of the expansion in Eq. (1.4).

Secondly, the infinitesimal small fluctuations of the fermionic fields around the background are [41]

$$\begin{aligned} \phi_\mu{}^A(x,y) &= \eta_i^A(y)\phi_\mu{}^i(x) + \ldots, \\ \chi^{ABC}(x,y) &= \eta_i^A(y)\eta_j^B(y)\eta_k^C(y)\chi^{ijk}(x) + \ldots \end{aligned} \tag{4.13}$$

Here, the orthonormal Killing spinors $\eta_i^A(y)$ transform curved SU(8) indices A, B, C into flat SU(8) indices i, j, k. Note that also the transformation parameter $\epsilon_A(x, y)$ and the related fermionic fields $X_\mu{}^{AB}(\epsilon)$ and $\Sigma_{ABCD}(\epsilon)$ are expanded accordingly [41],

$$\epsilon_A(x, y) = \eta_A^i(y)\epsilon_i(x) + \ldots, \tag{4.14}$$

$$X_\mu{}^{AB}(x, y) = \eta_i^A(y)\eta_j^B(y)X_\mu{}^{ij}(x) + \ldots, \tag{4.15}$$

$$\Sigma_{ABCD}(x, y) = \eta_i^A(y)\eta_j^B(y)\eta_k^C(y)\eta_l^D(y)\Sigma_{ijkl}(x) + \ldots \tag{4.16}$$

With these expansions, the supersymmetry transformations of the vierbeine in Eqs. (2.15, 3.41) are consistent with respect to Eq. (4.12). Since this thesis investigates the bosonic uplift of $N = 8$ supergravity to 11 dimensions, it is not necessary to go into more details here.

Let us now expand the vector fields $\mathcal{B}_\mu{}^{\mathcal{M}}(x, y)$ of 11-dimensional supergravity around the background. The correct relation *to all orders* takes the form[4]

$$\mathcal{B}_\mu{}^{\mathcal{M}}(x, y) = \mathcal{R}^{\mathcal{M}}{}_{\mathcal{N}}(y) A_\mu{}^{\mathcal{N}}(x), \tag{4.17}$$

where the $E_{7(7)}$ rotation matrix $\mathcal{R}^{\mathcal{M}}{}_{\mathcal{N}}(y)$ has been found in Refs. [45, 47, 49]. In principle, it rotates the SO(8) indices of the four-dimensional gauge bosons into the internal seven-dimensional indices of the 11-dimensional vector fields. More explicitly, the upper index $\mathcal{M}$ of the transformation matrix $\mathcal{R}^{\mathcal{M}}{}_{\mathcal{N}}(y)$ is decomposed under GL(7,$\mathbb{R}$) (Eq. (3.28)), whereas the lower index $\mathcal{N}$ is decomposed under SL(8,$\mathbb{R}$) (Eq. (2.5)),

$$\mathcal{R}^{\mathcal{M}}{}_{\mathcal{N}} = \begin{pmatrix} \mathcal{R}^m{}_{IJ} & \mathcal{R}^{m\,IJ} \\ \mathcal{R}^{mn}{}_{IJ} & \mathcal{R}^{mn\,IJ} \\ \mathcal{R}_{mn\,IJ} & \mathcal{R}_{mn}{}^{IJ} \\ \mathcal{R}_{m\,IJ} & \mathcal{R}_m{}^{IJ} \end{pmatrix}. \tag{4.18}$$

The non-zero components are

$$\mathcal{R}^m{}_{IJ} = \frac{\sqrt{2}}{8} K^{m\,IJ}, \qquad \mathcal{R}^{mn}{}_{IJ} = \frac{\sqrt{2}}{8}\left(\mathring{\zeta}^m K^{n\,IJ} - \mathring{\zeta}^n K^{m\,IJ} - K^{mn\,IJ}\right), \tag{4.19}$$

$$\mathcal{R}_{mn}{}^{IJ} = -\frac{\sqrt{2}}{8} K_{mn}{}^{IJ}, \qquad \mathcal{R}_m{}^{IJ} = -\frac{\sqrt{2}}{8}\left(\mathring{\zeta}^n K_{mn}{}^{IJ} - K_m{}^{IJ}\right). \tag{4.20}$$

They depend on the Killing vectors $K_m{}^{IJ}(y)$ and -forms $K_{mn}{}^{IJ}(y)$ as well as on the (seven-dimensional) dual volume potential $\mathring{\zeta}^m(y)$ of the seven-sphere. The latter is defined as

$$\mathring{\zeta}^n = 6\,\mathring{\eta}^{nm_1\cdots m_6}\mathring{\zeta}_{m_1\cdots m_6}, \qquad \mathring{\zeta}_{m_1\cdots m_6} = \frac{1}{6\cdot 6!}\,\mathring{\eta}_{m_1\cdots m_7}\,\mathring{\zeta}^{m_7}. \tag{4.21}$$

[4]The last seven components of $\mathcal{B}_\mu{}^{\mathcal{M}}$ belong to the non-physical dual gravitons. Hence, Eq. (4.17) only makes sense for the first 49 components.

Note the non-standard normalization of $\mathring{\zeta}^m$ for later convenience.

Here is a simple example: The first seven components of $\mathcal{B}_\mu{}^{\mathcal{M}}(x,y)$ are proportional to the vectors $B_\mu{}^m(x,y)$ that parametrize the upper off-diagonal of the elfbein (Eq. (3.13)). With Eqs. (4.17, 4.19) one then finds the old Ansatz for the vector fields in Kaluza-Klein theory [29], i.e.

$$B_\mu{}^m(x,y) = -\frac{\sqrt{2}}{4} K^{m\,IJ}(y) A_\mu{}^{IJ}(x). \tag{4.22}$$

Finally, the correct expansion of the scalar 56-bein $\mathcal{V}^{\mathcal{M}}{}_{AB}(x,y)$ in 11 dimensions can be found by considering the respective supersymmetry transformations of the vectors (Eqs. (2.22, 3.42)) [45, 47]. The resulting relation between the reformulated 11-dimensional and the four-dimensional scalars is quite similar to the vector relation,

$$\mathcal{V}^{\mathcal{M}}{}_{AB}(x,y) = \sqrt{2}\,\mathcal{R}^{\mathcal{M}}{}_{\mathcal{N}}(y)\,\eta^i_A(y)\,\eta^j_B(y)\,\hat{\mathcal{V}}^{\mathcal{N}}{}_{ij}(x). \tag{4.23}$$

However, it contains the necessary rotations in order to rotate the curved SU(8) indices A, B into flat ones i, j. For later convenience, one computes the components of

$$\mathcal{V}^{\mathcal{M}}{}_{ij}(x,y) = \sqrt{2}\mathcal{R}^{\mathcal{M}}{}_{\mathcal{N}}(y)\hat{\mathcal{V}}^{\mathcal{N}}{}_{ij}(x) \tag{4.24}$$

using Eqs. (4.19, 4.20). In particular,

$$\mathcal{V}^m{}_{ij} = \frac{\sqrt{2}i}{8} K^{m\,IJ}\left(u_{ij}{}^{IJ} + v_{ij\,IJ}\right), \tag{4.25}$$

$$\mathcal{V}^{mn}{}_{ij} = \frac{\sqrt{2}i}{8}\left(\mathring{\zeta}^m K^{n\,IJ} - \mathring{\zeta}^n K^{m\,IJ} - K^{mn\,IJ}\right)\left(u_{ij}{}^{IJ} + v_{ij\,IJ}\right), \tag{4.26}$$

$$\mathcal{V}_{mn\,ij} = \frac{\sqrt{2}}{8} K_{mn}{}^{IJ}\left(u_{ij}{}^{IJ} - v_{ij\,IJ}\right), \tag{4.27}$$

$$\mathcal{V}_{m\,ij} = \frac{\sqrt{2}}{8}\left(\mathring{\zeta}^n K_{mn}{}^{IJ} - K_m{}^{IJ}\right)\left(u_{ij}{}^{IJ} - v_{ij\,IJ}\right). \tag{4.28}$$

Again, the signs are adapted in comparison to Refs. [48, 49], because the components above constitute the 56-bein with an *upper* $\mathrm{E}_{7(7)}$ index, $\hat{\mathcal{V}}^{\mathcal{M}}{}_{ij}$.

4.3. Bosonic Uplift Ansätze

Starting with the relations for the reformulated 11-dimensional fields in Eqs. (4.17, 4.23), one may now find explicit non-linear uplift formulae for the bosonic fields of 11-dimensional supergravity in terms of the four-dimensional fields:

$$\begin{aligned}
\left(B_\mu{}^m,\;\; A_{\mu mn},\;\; A_{\mu m_1\cdots m_5},\;\; A_{\mu m_1\cdots m_7|n}\right)(x,y) \quad &\Leftrightarrow \quad \left(A_\mu{}^{IJ},\;\; A_{\mu\,IJ}\right)(x),\\
\left(g_{mn},\;\; A_{mnp},\;\; A_{m_1\cdots m_6}\right)(x,y) \quad &\Leftrightarrow \quad \left(u_{ij}{}^{IJ},\;\; v_{ij\,IJ}\right)(x).
\end{aligned}$$

As it turns out, only the scalar uplift Ansätze are non-trivial. Indeed, let us assume for a moment that they are already known. Then, the vector uplift relations can easily be found in the following way: First, the vectors $B_\mu{}^m(x,y)$ are given in Eq. (4.22). Secondly, the components 8 to 28 of the vector relation in Eq. (4.17) read

$$\mathcal{B}_{\mu\, mn}(x,y) = \mathcal{R}_{mn}{}^{IJ}(y) A_{\mu\, IJ}(x).$$

Using Eqs. (3.24, 4.20), one then finds an explicit Ansatz for the 21 vectors $A_{\mu\, mn}(x,y)$, i.e.

$$A_{\mu\, mn}(x,y) = \frac{1}{4!} K_{mn}{}^{IJ}(y) A_{\mu\, IJ}(x) + B_\mu{}^p(x,y) A_{mnp}(x,y). \tag{4.29}$$

Indeed, if the scalar fields $A_{mnp}(x,y)$ are known, the rhs can be evaluated too. The Ansätze for $A_{\mu m_1\cdots m_5}(x,y)$ and $A_{\mu m_1\cdots m_7|n}(x,y)$ can be derived in the same iterative way (assuming the scalar Ansätze for $g_{mn}(x,y)$, $A_{mnp}(x,y)$ and $A_{m_1\cdots m_6}(x,y)$ are already known).

Let us now derive the necessary non-linear scalar uplift Ansätze. Therefore, the main problem of comparing the vielbein components of 11-dimensional and $N = 8$ supergravity is the occurrence of the SU(8) rotating Killing spinors in Eq. (4.23). However, these are orthonormal and drop out in non-linear SU(8) invariant combinations of the vielbeine.

For example, consider the expression

$$\mathcal{V}^m{}_{AB} \mathcal{V}^{n\, AB} = \eta^i_A \eta^j_B \mathcal{V}^m{}_{ij} \eta^A_k \eta^B_l \mathcal{V}^{n\, kl} = \mathcal{V}^m{}_{ij} \mathcal{V}^{n ij}.$$

Indeed, the Killing spinors $\eta^i_A(y)$ drop out. One now uses Eq. (3.23) on the lhs and Eq. (4.25) on the rhs, which results in an uplift Ansatz for the inverse metric scaled with the warp factor [43], i.e.

$$\Delta^{-1} g^{mn}(x,y) = \frac{1}{8} K^{m\, IJ}(y) K^{n\, KL}(y) \left(u_{ij}{}^{IJ} + v_{ij\, IJ}\right)\left(u^{ij}{}_{KL} + v^{ij\, KL}\right)(x). \tag{4.30}$$

For the lhs, one used the Clifford algebra of the Γ matrices in Eq. (3.17). As expected, this relation is non-linear in the four-dimensional fields, which will also be the case for the following uplift Ansätze.

In a similar way, one relates

$$\mathcal{V}_{mn}{}^{AB} \mathcal{V}^p{}_{AB} = \mathcal{V}_{mn}{}^{ij} \mathcal{V}^p{}_{ij},$$

which yields a non-linear uplift Ansatz for the internal three-form [45, 47]. Indeed, using Eqs. (3.23, 3.26) on the lhs as well as Eqs. (4.25, 4.27) on the rhs, one finds

$$\Delta^{-1} A_{mn}{}^p(x,y) = -\frac{\sqrt{2}i}{96} K_{mn}{}^{IJ}(y) K^{p\, KL}(y) \left(u^{ij}{}_{IJ} - v^{ij\, IJ}\right)\left(u_{ij}{}^{KL} + v_{ij\, KL}\right)(x). \tag{4.31}$$

In order to derive an uplift Ansatz for the internal six-form potential $A_{m_1\cdots m_6}(x,y)$, one introduces the (seven-dimensional) dual one-form

$$A^n = 6\,\epsilon^{nm_1\cdots m_6} A_{m_1\cdots m_6}. \tag{4.32}$$

Similar to the dual volume potential on the round seven-sphere, $\mathring{\zeta}^m(y)$, a non-standard normalization is used for later convenience. The internal six-form potential is a tensor and its (seven-dimensional) dual $A^n(x,y)$ is constructed with the full ϵ tensor. However, one can convert this ϵ tensor to the tensor density $\mathring{\eta}\,(=\pm 1, 0)$ using the internal seven-bein $e_m{}^a(x,y)$ and the definition of the warp factor in Eq. (3.16),

$$\epsilon_{m_1\cdots m_7} = e_{m_1}{}^{a_1}\ldots e_{m_7}{}^{a_7}\mathring{\eta}_{a_1\cdots a_7} = \Delta\mathring{\eta}_{m_1\cdots m_7}. \tag{4.33}$$

Eq. (4.32) then reads

$$A^n = \frac{6}{\Delta}\mathring{\eta}^{nm_1\cdots m_6}A_{m_1\cdots m_6} \qquad \Leftrightarrow \qquad A_{m_1\cdots m_6} = \frac{\Delta}{6\cdot 6!}\mathring{\eta}_{m_1\cdots m_7}A^{m_7}. \tag{4.34}$$

Note that the indices of the six-form potential and its dual are raised and lowered with the full internal metric.

Now, let us consider the relation

$$\mathcal{V}^{mn}{}_{AB}\mathcal{V}^{pAB} = \mathcal{V}^{mn}{}_{ij}\mathcal{V}^{pij}$$

and insert the various vielbein components in Eqs. (3.23, 3.27) and Eqs. (4.25, 4.26). This gives an equation for $A^n(x,y)$, i.e.

$$\begin{aligned}\frac{\sqrt{2}}{9}\left(\Delta A^{[m} + 3\sqrt{2}\mathring{\zeta}^{[m}\right) g^{n]p} = \mathring{\eta}^{mnq_1\cdots q_5}A^p{}_{q_1q_2}A_{q_3q_4q_5} + \frac{\Delta}{24}K^{mn\,IJ}K^{p\,KL} \\ \times\left(u_{ij}{}^{IJ} + v_{ij\,IJ}\right)\left(u^{ij}{}_{KL} + v^{ij\,KL}\right).\end{aligned} \tag{4.35}$$

When contracting this relation with g_{np}, the first term on the rhs drops out because

$$A_{[mnp}A_{qrs]} = 0.$$

In particular,

$$\Delta A^m + 3\sqrt{2}\mathring{\zeta}^m = \frac{\Delta}{8\sqrt{2}}\,g_{np}\,K^{mn\,IJ}K^{p\,KL}\left(u_{ij}{}^{IJ} + v_{ij\,IJ}\right)\left(u^{ij}{}_{KL} + v^{ij\,KL}\right), \tag{4.36}$$

and dualizing this expression using Eq. (4.21, 4.34) yields

$$A_{m_1\cdots m_6} + 3\sqrt{2}\mathring{\zeta}_{m_1\cdots m_6} = \frac{\sqrt{2}}{96\cdot 6!}\,\epsilon_{nm_1\cdots m_6}\,g_{pq}K^{np\,IJ}K^{q\,KL}\left(u_{ij}{}^{IJ} + v_{ij\,IJ}\right)\left(u^{ij}{}_{KL} + v^{ij\,KL}\right). \tag{4.37}$$

The rhs of Eqs. (4.36, 4.37) further simplify using the uplift Ansatz for the inverse metric in Eq. (4.30) and the definition of the Killing two-form in Eq. (4.9). They are proportional to

$$\mathring{D}^m \log \Delta = \Delta^{-1} \mathring{D}^m \Delta = \frac{1}{2} g^{pq} \mathring{D}^m g_{pq}, \tag{4.38}$$

which finally gives a simpler non-linear Ansatz for the six-form potential, i.e.

$$\Delta A^m(x,y) + 3\sqrt{2}\mathring{\zeta}^m(y) = \frac{9\sqrt{2}}{4m_7} \mathring{D}^m \log \Delta(x,y), \tag{4.39}$$

$$A_{m_1 \cdots m_6}(x,y) + 3\sqrt{2}\mathring{\zeta}_{m_1 \cdots m_6}(y) = \frac{\sqrt{2}}{16 \cdot 5! m_7} \mathring{\eta}_{m_1 \cdots m_7} \mathring{D}^{m_7} \log \Delta(x,y). \tag{4.40}$$

This result has already been derived in Ref. [48]. In comparison to Eqs. (4.36, 4.37), the Ansätze in Eqs. (4.39, 4.40) do not require the metric g_{mn} but an explicit expression for the warp factor.

The formula for the inverse metric in Eq. (4.30) has been used to construct several gaugings of 11-dimensional supergravity, for example the G_2, SO(3)×SO(3) and SU(3)×U(1)×U(1) invariant solutions [43, 51, 57, 58]. In such cases, the explicit expression for $\Delta^{-1} g^{mn}(x,y)$ has been inverted to obtain the metric $\Delta g_{mn}(x,y)$, and the warp factor could be removed by explicitly taking the determinant (Eq. 3.16). In this way, one could finally lower indices and derive the 11-dimensional scalar fields $A_{mnp}(x,y)$ and $A_{m_1 \cdots m_6}(x,y)$.

Let us now derive a new *direct* uplift Ansatz for the metric $g_{mn}(x,y)$ [49, 50], which also implies direct Ansätze for the form-potentials and the warp factor. First, consider the relation

$$\mathcal{V}_{mpAB} \mathcal{V}^p{}_{CD} \mathcal{V}_{nq}{}^{[AB} \mathcal{V}^{qCD]} = \mathcal{V}_{mpij} \mathcal{V}^p{}_{kl} \mathcal{V}_{nq}{}^{[ij} \mathcal{V}^{qkl]}.$$

One uses Eqs. (3.23, 3.26) on the lhs and simplifies all terms including a factor of A_{mnp} to

$$\ldots A_{mnp} \Gamma^n{}_{[AB} \Gamma^p{}_{CD]} \ldots = 0. \tag{4.41}$$

Such expressions vanish because an antisymmetric index pair $[np]$ is contracted with a symmetric index pair (np). Using Eq. (A.7), one then finds the metric on the lhs,

$$\Delta^{-2} g_{mn} = \frac{16}{3} \mathcal{V}_{mp\,ij} \mathcal{V}^p{}_{kl} \mathcal{V}_{nq}{}^{[ij} \mathcal{V}^{q\,kl]}.$$

For the rhs, one uses Eqs. (4.25, 4.27) and obtains[5]

$$\mathcal{V}_{mp\,[ij} \mathcal{V}^p{}_{kl]} = \frac{i}{32} K_{mp}{}^{IJ} K^{p\,KL} \left(u_{[ij}{}^{IJ} - v_{[ij\,IJ} \right) \left(u_{kl]}{}^{KL} + v_{kl]\,KL} \right).$$

[5] This equation as well as Eqs. (4.42, 4.51) differ from the respective expressions in Ref. [49] by a sign. This is due to the sign difference in the vielbein components. However, the resulting expressions (for the metric and the four-form field-strength) in terms of the four-dimensional scalar fields $u_{ij}{}^{IJ}(x)$ and $v_{ij\,IJ}(x)$ remain uneffected.

These two equations together represent a useful metric Ansatz in terms of the Killing forms and the four-dimensional scalar fields,

$$\Delta^{-2} g_{mn} = -\frac{1}{192} K_{mp}{}^{IJ} K^{p\,KL} K_{nq}{}^{MN} K^{q\,PQ} \left(u_{[ij}{}^{IJ} - v_{[ij\,IJ}\right) \times \left(u_{kl]}{}^{KL} + v_{kl]\,KL}\right) \left(u^{ij}{}_{MN} - v^{ij\,MN}\right) \left(u^{kl}{}_{PQ} + v^{kl\,KL}\right).$$

However, one may simplify the resulting expression further: Using Eqs. (A.16, A.17) in Appendix A yields

$$\mathcal{V}_{mp[ij}\mathcal{V}^{p}{}_{kl]} = \frac{i}{8}\left(\mathcal{A}_{mijkl} - \mathcal{B}_{mijkl}\right), \tag{4.42}$$

where the convenient tensors $\mathcal{A}_{mijkl}(x,y)$ and $\mathcal{B}_{mijkl}(x,y)$ are defined as

$$\mathcal{A}_{mijkl}(x,y) = \frac{1}{4} K_{mn}{}^{[IJ}(y) K^{n\,KL]}(y) \left(u_{ij}{}^{IJ} u_{kl}{}^{KL} - v_{ij\,IJ} v_{kl\,KL}\right)(x), \tag{4.43}$$

$$\mathcal{B}_{mijkl}(x,y) = K_{m}{}^{IJ}(y) \left(u_{ij}{}^{IK} v_{kl\,JK} - v_{ij\,IK} u_{kl}{}^{JK}\right)(x). \tag{4.44}$$

By definition, these are totally antisymmetric in the SU(8) indices $[ijkl]$ and depend on all 11 coordinates (x,y). In terms of these tensors, the metric Ansatz finally reads

$$\Delta^{-2} g_{mn} = \frac{1}{12}\left(\mathcal{A}_{mijkl} - \mathcal{B}_{mijkl}\right)\left(\mathcal{A}_{n}{}^{ijkl} - \mathcal{B}_{n}{}^{ijkl}\right). \tag{4.45}$$

Note that this Ansatz is quartic in the four-dimensional scalar fields $u_{ij}{}^{IJ}(x)$ and $v_{ij\,IJ}(x)$, whereas the Ansätze for the inverse metric and the mixed three-form potential were only quadratic.

Let us combine the Ansätze for the metric and its inverse in Eqs. (4.45, 4.30) to get a new Ansatz for the warp factor $\Delta(x,y)$. This can be done because the new metric Ansatz contains a proportionality factor of Δ^{-2}. One finds

$$\Delta^{-3} = \frac{1}{28\cdot 4!}\,\mathcal{C}_{ij}{}^{klmn}\mathcal{C}^{ij}{}_{klmn}, \tag{4.46}$$

where the $\mathcal{C}(x,y)$ tensor is defined as

$$\mathcal{C}_{pq}{}^{ijkl} = K^{mIJ}\left(u_{pq}{}^{IJ} + v_{pq\,IJ}\right)\left(\mathcal{A}_{m}{}^{ijkl} - \mathcal{B}_{m}{}^{ijkl}\right). \tag{4.47}$$

Similarly, one combines the Ansatz for the three-form with mixed index structure in Eq. (4.31) and the metric Ansatz in Eq. (4.45) to obtain a new Ansatz for the full internal three-form potential,

$$A_{mnp} = -\frac{16\sqrt{2}}{9}\Delta^{3}\,\mathcal{V}_{mn}{}^{AB}\mathcal{V}_{pq}{}^{[CD}\mathcal{V}^{q\,EF]}\mathcal{V}^{r}{}_{AB}\mathcal{V}_{rs\,CD}\mathcal{V}^{s}{}_{EF}.$$

Using Eq. (4.41, A.11), one has

$$\mathcal{V}_{pq}{}^{[CD}\mathcal{V}^{q\,EF]} = \frac{1}{2}\left(\mathcal{V}_{pq}{}^{CD}\mathcal{V}^{q\,EF} + \mathcal{V}_{pq}{}^{EF}\mathcal{V}^{q\,CD}\right),$$

and replacing the curved SU(8) indices by flat SU(8) indices yields

$$A_{mnp} = -\frac{8\sqrt{2}}{9}\Delta^3\,\mathcal{V}_{mn}{}^{i_1i_2}\left(\mathcal{V}_{pq}{}^{i_3i_4}\mathcal{V}^{q\,i_5i_6} + \mathcal{V}_{pq}{}^{i_5i_6}\mathcal{V}^{q\,i_3i_4}\right)\mathcal{V}^{r}{}_{i_1i_2}\mathcal{V}_{rs\,i_3i_4}\mathcal{V}^{s}{}_{i_5i_6}. \tag{4.48}$$

With a final look at Eqs. (4.25, 4.27), one finally finds

$$A_{mnp} = -\frac{\sqrt{2}i}{48\cdot 4!}\Delta^3 K_{mn}{}^{IJ}\left(u^{ij}{}_{IJ} - v^{ij\,IJ}\right)\mathcal{C}_{ij}{}^{qrst}\left(\mathcal{A}_{p\,qrst} - \mathcal{B}_{p\,qrst}\right). \tag{4.49}$$

The Ansätze for the warp factor and the full three-form potential in Eqs. (4.46, 4.49) are derived by combining the Ansätze for $\Delta^{-1}g^{mn}(x,y)$, $\Delta^{-1}A_{mn}{}^{p}(x,y)$ and $\Delta^{-2}g_{mn}(x,y)$ in Eqs. (4.30, 4.31, 4.45). Hence, they are sextic in the scalar fields $u_{ij}{}^{IJ}(x)$ and $v_{ij\,IJ}(x)$ and not suitable for the construction of the group invariant solutions in the next chapters. Therefore, it is more convenient to use Eqs. (4.30, 4.31, 4.45) to derive the explicit expressions for the metric, its inverse and the three-form with mixed index structure. In a second step, these can then be combined to obtain the warp factor and the full three-form potential.

As a final remark, the three-form Ansatz is not manifestly antisymmetric, which may be a hint that it can be simplified further using the $E_{7(7)}$ properties of the $u_{ij}{}^{IJ}(x)$ and $v_{ij\,IJ}(x)$ tensors in Section 2.3 [37, 41]. One such simplification concerns the $\mathcal{C}$ tensor that occurs in both the warp factor and the three-form potential. In particular, Appendix B shows that one may extract a Kronecker-delta out of it. However, this is not sufficient to show the explicit antisymmetry of $A_{mnp}(x,y)$.

4.4. Ansätze for the Internal Field-Strength and the Freund-Rubin Term

The scalar Ansätze derived so far are sufficient to *construct* an 11-dimensional supergravity solution. In particular, they are used in the next three chapters to find the explicit G_2 and SO(3)×SO(3) invariant solutions. In order to check the consistency, it will also be necessary to compute the internal four-form field-strength $F_{mnpq}(x,y)$ and the Freund-Rubin term $\mathfrak{f}_{\mathrm{FR}}(x,y)$. Therefore, this section presents the corresponding embedding formulae.

Let us start to find an Ansatz for $F_{mnpq}(x,y)$ [49, 50]. It is given by the formula

$$F_{mnpq} = 4!\,\mathring{D}_{[m}A_{npq]} \tag{4.50}$$

and the explicit three-form Ansatz in Eq. (4.48). In the resulting expression,

$$F_{mnpq} = -\frac{64\sqrt{2}}{3}\,\mathring{D}_{[m}\left(\Delta^3\,\mathcal{V}_{np}{}^{i_1i_2}\left(\mathcal{V}_{q]r}{}^{i_3i_4}\mathcal{V}^{r\,i_5i_6} + \mathcal{V}_{q]r}{}^{i_5i_6}\mathcal{V}^{r\,i_3i_4}\right)\mathcal{V}^{s}{}_{i_1i_2}\mathcal{V}_{st\,i_3i_4}\mathcal{V}^{t}{}_{i_5i_6}\right), \tag{4.51}$$

one needs to evaluate the derivative in general. First, one has

$$\mathring{D}_m\Delta^3 = 3\Delta^3\mathring{D}_m\log\Delta,$$

hence, one term in $F_{mnpq}(x,y)$ will be proportional to $A_{[mnp}\mathring{D}_{q]}\log\Delta$. Secondly, the covariant background derivative $\mathring{D}_m$ only acts on the y-dependent fields in the vielbein components: the Killing forms and the dual volume potential $\mathring{\zeta}^m(y)$. It does not act on the four-dimensional scalars $u_{ij}{}^{IJ}(x)$ and $v_{ij\,IJ}(x)$. In general, for the S^7 reduction, one uses Eq. (4.25 – 4.28) as well as Eq. (4.11) to find

$$m_7^{-1}\mathring{D}_m\mathcal{V}^n{}_{ij} = \mathring{g}_{mp}\left(2\mathring{\zeta}^{[n}\mathcal{V}^{p]}{}_{ij} - \mathcal{V}^{np}{}_{ij}\right), \tag{4.52}$$

$$m_7^{-1}\mathring{D}_m\mathcal{V}^{np}{}_{ij} = -2\left(\delta_m{}^{[n} + \mathring{\zeta}_m\mathring{\zeta}^{[n} - m_7^{-1}\mathring{D}_m\mathring{\zeta}^{[n}\right)\mathcal{V}^{p]}{}_{ij} - 2\mathring{g}_{mq}\mathring{\zeta}^{[n}\mathcal{V}^{p]q}{}_{ij}, \tag{4.53}$$

$$m_7^{-1}\mathring{D}_m\mathcal{V}_{np\,ij} = 2\mathring{g}_{m[n}\left(-\mathcal{V}_{p]\,ij} + \mathring{\zeta}^q\,\mathcal{V}_{p]q\,ij}\right), \tag{4.54}$$

$$m_7^{-1}\mathring{D}_m\mathcal{V}_{n\,ij} = \left(\mathring{\zeta}_m\delta_n{}^p - \mathring{g}_{mn}\mathring{\zeta}^p\right)\mathcal{V}_{p\,ij} - \left(\delta_m{}^p + \mathring{\zeta}_m\mathring{\zeta}^p - m_7^{-1}\mathring{D}_m\mathring{\zeta}^p\right)\mathcal{V}_{np\,ij}. \tag{4.55}$$

Putting all this together, the resulting intermediate expression for F_{mnpq} becomes rather long and is not listed here. However, it should be clear that it contains the tensors $\mathring{g}_{mn}$, $\mathring{\zeta}^m$ as well as all four-dimensional vielbeine $\mathcal{V}^{\mathcal{M}}{}_{ij}$. Let us perform some simplifications: One starts with replacing the $\mathcal{V}^{\mathcal{M}}{}_{ij}$'s by the 11-dimensional vielbein components $\mathcal{V}^{\mathcal{M}}{}_{AB}$ since the SU(8) indices $i, j, \ldots$ are fully contracted in pairs. Using Eqs. (3.23, 3.26, 3.27, 3.32) then introduces the 11-dimensional fields (e.g. A_{mnp} and $A_{m_1\cdots m_6}$) and the SU(8) Γ matrices. With Eqs. (A.1) for the traces of products of Γ matrices, Eq. (4.51) reduces to

$$F_{mnpq} = -72A_{[mnp}\mathring{D}_{q]}\log\Delta + \frac{24}{\sqrt{2}}m_7\,A_{[mnp}\mathring{g}_{q]r}\left(\Delta A^r + 3\sqrt{2}\mathring{\zeta}^r\right) + \left[4m_7\mathring{g}_{mr_1}\mathring{\eta}^{r_1\cdots r_7}\left(g_{nr_2}g_{pr_3}g_{qr_4} - 18A_{npr_2}A_{qr_3r_4}\right)A_{r_5r_6r_7}\right]\Big|_{[mnpq]},$$

where $|_{[mnpq]}$ denotes antisymmetrized indices $[mnpq]$. Furthermore, one eliminates the second term by Eq. (4.39),

$$F_{mnpq} = -18A_{[mnp}\mathring{D}_{q]}\log\Delta + \left[4m_7\mathring{g}_{mr_1}\mathring{\eta}^{r_1\cdots r_7}\left(g_{nr_2}g_{pr_3}g_{qr_4} - 18A_{npr_2}A_{qr_3r_4}\right)A_{r_5r_6r_7}\right]\Big|_{[mnpq]},$$

and the term $\mathring{\eta}^{r_1\cdots r_7}A_{qr_3r_4}A_{r_5r_6r_7}$ can be replaced via Eq. (4.35). Together with Eq. (4.39), this cancels the term proportional to $\mathring{D}_m \log\Delta$. Finally, one turns the tensor density $\mathring{\eta}^{r_1\cdots r_7}$ into the tensor $\epsilon^{r_1\cdots r_7}$ (Eq. (4.33)) and obtains

$$F_{mnpq} = m_7\Delta\mathring{g}_{s[m}\Big[4\epsilon_{npq]r_1r_2r_3}{}^{s}A^{r_1r_2r_3} - 3g_{n|t|}A_{pq]r}K^{rs\,IJ} \times K^{t\,KL}\left(u_{ij}{}^{IJ} + v_{ij\,IJ}\right)\left(u^{ij}{}_{KL} + v^{ij\,KL}\right)\Big]. \tag{4.56}$$

This formula appears to be more feasible for practical tests than previous expressions found in Refs. [48, 50].

As a convenient consequence, it is not difficult to raise all indices with the inverse metric $g^{mn}(x,y)$. Therefore, one must keep in mind that the indices of the Killing forms and $\mathring{g}_{mn}(y)$ are raised with the background metric. All other tensors in Eq. (4.56) are covariant, hence

$$F^{mnpq} = m_7\Delta\mathring{g}_{st}g^{t[m}\Big(4\epsilon^{npq]r_1r_2r_3s}A_{r_1r_2r_3} - 3A^{np}{}_{r}K^{q]\,IJ} \times K^{rs\,KL}\left(u^{ij}{}_{IJ} + v^{ij\,IJ}\right)\left(u_{ij}{}^{KL} + v_{ij\,KL}\right)\Big). \tag{4.57}$$

Note the power of the last step: So far, the field-strength with upper indices has always been found by raising each lower index of $F_{mnpq}(x,y)$ with the explicit expression for the inverse metric $g^{mn}(x,y)$. In Ref. [51], this was one of the hardest tasks in verifying the SO(3)×SO(3) invariant solution of 11-dimensional supergravity. In this thesis, Chapters 5, 6 and 7 will make use of the simple Ansätze for $F_{mnpq}(x,y)$ and $F^{mnpq}(x,y)$ above in order to verify the G_2 and SO(3)×SO(3) invariant solutions of 11-dimensional supergravity.

The rest of this section presents a non-linear uplift Ansatz for the Freund-Rubin term $\mathfrak{f}_{\mathrm{FR}}(x,y)$ of 11-dimensional supergravity. It is defined as the four-dimensional dual to the external four-form field-strength $F_{\mu\nu\rho\sigma}(x,y)$. In other words,

$$F_{\mu\nu\rho\sigma}(x,y) = i\mathfrak{f}_{\mathrm{FR}}(x,y)\mathring{\eta}_{\mu\nu\rho\sigma}, \tag{4.58}$$

where $\mathring{\eta}_{\mu\nu\rho\sigma}$ denotes the four-dimensional Levi-Cevita tensor density [59]. In particular, for a Freund-Rubin compactification, the Freund-Rubin term $\mathfrak{f}_{\mathrm{FR}}(x,y) = \mathfrak{f}_{\mathrm{FR}}$ becomes a constant. In the following, we repeat the main steps of Ref. [48] to derive an expression for $\mathfrak{f}_{\mathrm{FR}}(x,y)$ in terms of the four-dimensional fields and the Killing forms on the seven-sphere.

The starting point is the generalized vielbein postulate for the 56-bein $\mathcal{V}^{\mathcal{M}}{}_{AB}(x,y)$ in 11-dimensions [46, 54, 48],

$$\mathbf{D}_m\mathcal{V}_{\mathcal{N}\,AB} + \mathcal{Q}_m{}^{C}{}_{[A}\mathcal{V}_{\mathcal{N}\,B]C} = \mathcal{P}_{m\,ABCD}\mathcal{V}_{\mathcal{N}}{}^{CD}, \tag{4.59}$$

with the following definitions: First, $\mathbf{D}_m$ denotes the covariant $E_{7(7)}$ derivative along the first seven directions (the physical directions),

$$\mathbf{D}_m \mathcal{V}_{\mathcal{N}\,AB} = \partial_m \mathcal{V}_{\mathcal{N}\,AB} - \mathbf{\Gamma}_{m\mathcal{N}}{}^{\mathcal{P}} \mathcal{V}_{\mathcal{P}\,AB}. \tag{4.60}$$

Here, $\mathbf{\Gamma}_{m\mathcal{N}}{}^{\mathcal{P}}(x,y)$ is the corresponding $E_{7(7)}$ generalized Christoffel connection [54]. Secondly, $\mathcal{Q}^{A}_{m\,B}(x,y)$ denotes the generalized SU(8) spin connection [55],

$$\mathcal{Q}^{A}_{mB} = -\frac{1}{2}\omega_{m\,ab}\Gamma^{ab}_{AB} + \frac{\sqrt{2}}{14} i\Delta^2 \mathfrak{f}_{\mathrm{FR}}\Gamma_{m\,AB} - \frac{\sqrt{2}}{48} F_{mnpq}\Gamma^{npq}_{AB}. \tag{4.61}$$

In particular, it depends on the SO(7) spin-connection $\omega_{m\,ab}(y)$, the Freund-Rubin term and the internal four-form field-strength. Let us not go further into details here, since in the subsequent steps, the term in Eq. (4.59) that is proportional to the SU(8) spin connection $\mathcal{Q}^{A}_{m\,B}(x,y)$ will drop out. Finally, the SU(8) tensor $\mathcal{P}_{m\,ABCD}(x,y)$ is the 'generalized non-metricity', which measures the failure of the 56-bein to be covariantly constant under the generalized covariant derivative (including the SU(8) spin connection). Its components are

$$\mathcal{P}_{m\,ABCD} = \frac{\sqrt{2}}{56} i\Delta^2 \mathfrak{f}_{\mathrm{FR}}\Gamma_{mn\,[AB}\Gamma^{n}{}_{CD]} + \frac{\sqrt{2}}{32} F_{mnpq}\Gamma^{n}{}_{[AB}\Gamma^{pq}{}_{CD]} \tag{4.62}$$

and also depend on the Freund-Rubin term and the internal four-form [55]. Note that $\mathcal{P}_{m\,ABCD}(x,y)$ is totally antisymmetric in the SU(8) indices — the first term in Eq. (4.62) is selfdual and the second one is anti-selfdual (see Eq. (A.4)).

The next step is to project out the non-metricity from the GVP in Eq. (4.59) using the orthonormality of the 56-bein $\mathcal{V}^{\mathcal{M}}{}_{AB}(x,y)$ (Eq. (3.35)),

$$\mathcal{P}_{m\,ABCD} = -i\mathcal{V}^{\mathcal{M}}{}_{CD}\mathbf{D}_m \mathcal{V}_{\mathcal{M}\,AB}, \tag{4.63}$$

and to compare it with the definition in Eq. (4.62). More explicitly, another projection onto the self-dual part yields

$$\frac{\sqrt{2}}{56} i\Delta^2 \mathfrak{f}_{\mathrm{FR}} \mathcal{V}_{mp}{}^{AB}\mathcal{V}^{p\,CD}\Gamma_{nq[AB}\Gamma^{q}{}_{CD]} = -i\mathcal{V}_{mp}{}^{AB}\mathcal{V}^{p\,CD}\mathcal{V}^{\mathcal{M}}{}_{CD}\mathbf{D}_n \mathcal{V}_{\mathcal{M}\,AB}.$$

On the lhs, the term proportional to the field-strength $F_{mnpq}(x,y)$ drops out as it is anti-selfdual and $\mathcal{V}_{np}{}^{[AB}\mathcal{V}^{p\,CD]}$ is selfdual (see Eqs. (4.41, A.4)). With a slight look to the derivation of the metric Ansatz in Section 4.3, one finds that the term on the lhs is proportional to $g_{mn}(x,y)$. More explicitly,

$$\frac{3\sqrt{2}}{28}\mathfrak{f}_{\mathrm{FR}}\Delta g_{mn} = \mathcal{V}_{mp}{}^{AB}\mathcal{V}^{p\,CD}\mathcal{V}^{\mathcal{M}}{}_{CD}\mathbf{D}_n \mathcal{V}_{\mathcal{M}\,AB}.$$

Finally, the curved SU(8) indices are contracted in pairs and can therefore be replaced by flat ones. From Eq. (4.63), it is also clear that the second part on the rhs is by construction, totally antisymmetric in the flat SU(8) indices. Contracting with the inverse metric then yields

$$\mathfrak{f}_{\mathrm{FR}} = \frac{4}{3\sqrt{2}}\Delta^{-1}g^{mn}\mathcal{V}_{mp}{}^{ij}\mathcal{V}^{p\,kl}\mathcal{V}^{\mathcal{M}}{}_{[kl}\mathbf{D}_n\mathcal{V}_{\mathcal{M}\,ij]}.$$

Furthermore, using Eq. (4.42) and the concrete $\mathrm{E}_{7(7)}$ connection components [48] gives

$$\mathfrak{f}_{\mathrm{FR}} = \frac{m_7}{42\sqrt{2}}\Delta^{-1}g^{mn}\left(\mathcal{A}_m{}^{ijkl} - \mathcal{B}_m{}^{ijkl}\right)\left(3\mathcal{A}_{n\,ijkl} + 4\mathcal{B}_{n\,ijkl}\right),$$

which reduces to

$$\mathfrak{f}_{\mathrm{FR}} = \frac{m_7}{\sqrt{2}}\left[\frac{\Delta^{-1}}{12}g^{mn}\left(\mathcal{A}_m{}^{ijkl} - \mathcal{B}_m{}^{ijkl}\right)\left(\mathcal{A}_{n\,ijkl} + \mathcal{B}_{n\,ijkl}\right) - \Delta^{-3}\right], \qquad (4.64)$$

when subtracting the explicit metric Ansatz in Eq. (4.45).

In Refs. [44, 48], similar Ansätze have been used to find the explicit form of the Freund-Rubin term for different group invariant solutions of 11-dimensional supergravity. It turned out that in all these cases, $\mathfrak{f}_{\mathrm{FR}}(x, y)$ consists of two parts. The first one is proportional to the corresponding scalar potential $V(x)$ of the four-dimensional theory and does not depend on the internal coordinates. The second one is proportional to the first variation of $V(x)$, the proportionality factor being y-dependent. In particular, this second part vanishes at stationary points of the scalar potential and the Freund-Rubin term becomes independent of the internal coordinates [44].

These considerations led to the following conjecture for the Freund-Rubin term: [48]

$$\mathfrak{f}_{\mathrm{FR}}(x, y) = \frac{m_7}{\sqrt{2}g^2}\left(-V(x) + \frac{g^2}{24}\left(Q^{ijkl}(x)\hat{\Sigma}_{ijkl}(x, y) + \text{h.c.}\right)\right). \qquad (4.65)$$

Here, the scalar potential $V(x)$ and the Q tensor are given in Eqs. (2.13, 2.25) and the complex-selfdual tensor $\hat{\Sigma}_{ijkl}(x, y)$ is defined as

$$\hat{\Sigma}_{ijkl}(x, y) = \left(u_{ij}{}^{IJ}u_{kl}{}^{KL} - v_{ij\,IJ}v_{kl\,KL}\right)(x)K_m{}^{[IJ}(y)K^{m\,KL]}(y). \qquad (4.66)$$

Since $Q^{ijkl}(x)$ is by construction complex anti-selfdual at stationary points, the above conjecture reflects the observed properties of the Freund-Rubin term. Unfortunately, Eq. (4.65) has not been proven in full generality yet. We proved the conjecture up to quadratic order in the scalar expectation value and also showed that it explicitly holds for the G_2 invariant solution of 11-dimensional supergravity [48].

This thesis does not go into further details here. In Chapters 6 and 7, the Freund-Rubin term will be explicitly computed for the G_2 and $\mathrm{SO}(3)\times\mathrm{SO}(3)$ invariant solutions of 11-dimensional supergravity using the uplift Ansatz in Eq. (4.64). As it turns out, this is much more convenient than using the conjecture in Eq. (4.65).

5. How To: Find a Group Invariant Solution of 11-Dimensional Supergravity

This chapter gives a general overview to the application of the derived scalar uplift Ansätze for certain gaugings of the $N = 8$ supergravity. In such cases, the obtained 11-dimensional fields can be written in terms of certain group invariant tensors, which are adapted to the corresponding deformed S^7 geometry. These fields then constitute a *group invariant solution* of 11-dimensional supergravity. Since this thesis does not give the fermionic uplift Ansätze, the most general solution that one may construct here is purely bosonic (by setting all fermions to zero).

A special case is the class of 'Freund-Rubin solutions with flux' [59]. These can be found by restricting the spacetime $\mathcal{M}_4$ in Eq. (4.1) to be maximally symmetric (here, AdS_4). In such cases, all fermions and vector fields must vanish identically. The only non-vanishing fields are the AdS_4 spacetime metric $\mathring{g}_{\mu\nu}(x)$ and the scalar degrees of freedom that are in the internal components of the metric and the form potentials $g_{mn}(x, y)$, $A_{mnp}(y)$ and $A_{m_1\cdots m_6}(y)$. In particular, the elfbein and the 11-dimensional four-form field-strength are then given by

$$E_M{}^A(x,y) = \begin{pmatrix} \Delta^{-1/2}(y)\mathring{e}_\mu{}^\alpha(x) & 0 \\ 0 & e_m{}^a(y) \end{pmatrix}, \quad F_{MNPQ} = \begin{cases} F_{\mu\nu\rho\sigma} = i\mathfrak{f}_{\mathrm{FR}}\mathring{\eta}_{\mu\nu\rho\sigma}(x) \\ F_{mnpq} = F_{mnpq}(y) \\ 0, \quad \text{otherwise} \end{cases} . \tag{5.1}$$

This simplification is only consistent when the derived 11-dimensional fields are evaluated at the group invariant stationary point of the scalar potential $V(x)$ (Eq. (2.13)). In this case, the Freund-Rubin term becomes a constant $\mathfrak{f}_{\mathrm{FR}}(x, y) = \mathfrak{f}_{\mathrm{FR}}$.

Note that for a Freund-Rubin solution with flux, *all* required uplift Ansätze are derived in the previous Chapter. However, for a certain gauging of the $N = 8$ supergravity, they may still be simplified. In particular, Section 5.1 now shows how the 11-dimensional fields can be written solely in terms of certain group invariant tensors. In principle, further consistency checks of the obtained solution are not required, since the uplift formulae have been found by a careful analysis of the supersymmetry transformations in both theories. However, for the readers convenience, Section 5.2 summarizes two kinds of consistency checks that will be performed in the next chapters for the found G_2 and $SO(3)\times SO(3)$ invariant solutions of 11-dimensional supergravity.

5.1. Using the Uplift Formulae

This section gives the general guideline to derive a certain *group invariant* solution of 11-dimensional supergravity. Therefore, one starts with the uplift Ansätze derived in the previous chapter, which are explicit relations between the following fields:[1]

$$\Big(g_{mn},\quad A_{mnp},\quad A_{m_1\cdots m_6},\quad \Delta,\quad F_{mnpq},\quad \mathfrak{f}_{\mathrm{FR}},\Big)\,(x,y)$$
$$\Leftrightarrow$$
$$\Big(u_{ij}{}^{IJ}(x),\quad v_{ij\,IJ}(x),\quad K_m{}^{IJ}(y),\quad K_{mn}{}^{IJ}(y),\quad \mathring{\zeta}^m(y)\Big)\,.$$

Based on the group invariance, it is now possible to simplify these Ansätze. In general, the larger the symmetry group, the more simplifications are possible. Let us now summarize the steps that lead to these simplifications.

One first writes the $u_{ij}{}^{IJ}(x)$ and $v_{ij\,IJ}(x)$ tensors in terms of the scalar and pseudoscalar vacuum expectation value $\phi_{IJKL}(x)$. Therefore, the four-dimensional 56-bein in Eq. (2.4) that encodes the four-dimensional scalars may be brought into unitary gauge, such that

$$\mathcal{V} = \exp\begin{pmatrix} 0 & \phi_{IJKL} \\ \phi^{IJKL} & 0 \end{pmatrix}. \tag{5.2}$$

Here, the scalar vacuum expectation value is a complex, selfdual tensor field,

$$\phi^{IJKL} = \phi^*_{IJKL} = \frac{1}{24}\epsilon^{IJKLMNPQ}\phi_{MNPQ}. \tag{5.3}$$

In this gauge, there is no distinction between SU(8) indices $ij\ldots$ and SL(8,$\mathbb{R}$) indices $IJ\ldots$ — they are all SO(8) indices now. Comparing with the unitary gauge of the 56-bein in Eq. (2.4) yields the useful relations[2]

$$u_{IJ}{}^{KL} = \sum_{n=0}^{\infty}\frac{1}{(2n)!}\left[(\phi\phi^*)^n\right]_{IJKL}\,,\qquad v^{IJKL} = \sum_{n=0}^{\infty}\frac{1}{(2n+1)!}\left[\phi^*(\phi\phi^*)^n\right]_{IJKL}\,. \tag{5.4}$$

[1]As discussed in Section 4.3, these formulae also imply the corresponding vector relations.
[2]Here and in the following, one uses the short-hand notation

$$A\,B = (A\,B)_{IJKL} = A_{IJMN}B_{MNKL}.$$

Taking this into account, the scalar uplift Ansätze can be written as

$$\Big(g_{mn}, \quad A_{mnp}, \quad A_{m_1\cdots m_6}, \quad \Delta, \quad F_{mnpq}, \quad \mathfrak{f}_{\rm FR},\Big)\,(x,y)$$
$$\Leftrightarrow$$
$$\Big(\phi_{IJKL}(x) \quad K_m{}^{IJ}(y), \quad K_{mn}{}^{IJ}(y), \quad \mathring{\zeta}^m(y)\Big)\,.$$

Secondly, for a group invariant solution of 11-dimensional supergravity, the most general form of the scalar vacuum expectation value must inherit the underlying symmetry. Hence,

$$\phi^{IJKL}(x) = \sum_r \lambda^{(r)}(x)\Phi^{(r)}_{IJKL} + i\sum_s \mu^{(s)}(x)\Psi^{(s)}_{IJKL}, \tag{5.5}$$

where $\left\{\Phi^{(r)}_{IJKL}\right\}$ and $\left\{\Psi^{(s)}_{IJKL}\right\}$ form a basis of *group invariant* real selfdual and real anti-selfdual 4-forms[3]. The coordinates $\left\{\lambda^{(r)}(x), \mu^{(s)}(x)\right\}$ parametrize the scalar manifold and take certain constant values at stationary points of the scalar potential $V(x)$. Some examples of invariant 4-forms are [51, 53]

$$\begin{aligned}
&\Phi^{(0)}_{IJKL} = C^+_{IJKL}\,, \quad \Psi^{(0)}_{IJKL} = 0 && \text{for SO}(7)^+ \text{ symmetry,}\\
&\Phi^{(0)}_{IJKL} = 0\,, \quad \Psi^{(0)}_{IJKL} = C^-_{IJKL} && \text{for SO}(7)^- \text{ symmetry,}\\
&\Phi^{(0)}_{IJKL} = C^+_{IJKL}\,, \quad \Psi^{(0)}_{IJKL} = C^-_{IJKL} && \text{for G}_2 \text{ symmetry,}\\
&\left.\begin{aligned}\Phi^{(0)}_{IJKL} = Y^+_{IJKL}\,, \quad \Psi^{(0)}_{IJKL} = Y^-_{IJKL}\\ \Phi^{(1)}_{IJKL} = Z^+_{IJKL}\,, \quad \Psi^{(1)}_{IJKL} = Z^-_{IJKL}\end{aligned}\right\} && \text{for SO}(3)\times\text{SO}(3) \text{ symmetry.}
\end{aligned}$$

The explicit expressions for the SO(8) tensors $C^{\pm}_{IJKL}$, $Y^{\pm}_{IJKL}$, $Z^{\pm}_{IJKL}$ will be given in the next two chapters. With these considerations, the scalar uplift Ansätze are simplified to

$$\Big(g_{mn}, \quad A_{mnp}, \quad A_{m_1\cdots m_6}, \quad \Delta \quad F_{mnpq}, \quad \mathfrak{f}_{\rm FR},\Big)\,(x,y)$$
$$\Leftrightarrow$$
$$\Big(\left\{\Phi^{(r)}_{IJKL}\right\}, \quad \left\{\Psi^{(s)}_{IJKL}\right\}, \quad \left\{\lambda^{(r)}(x)\right\}, \quad \left\{\mu^{(s)}(x)\right\}, \quad K_m{}^{IJ}(y), \quad K_{mn}{}^{IJ}(y), \quad \mathring{\zeta}^m(y)\Big)\,.$$

To work with a group invariant scalar field configuration (Eq. (5.5)) has another convenient side effect: Higher-order products of $\left\{\Phi^{(r)}_{IJKL}\right\}$ and $\left\{\Psi^{(s)}_{IJKL}\right\}$, such as

$$\Phi^{(r)}\Phi^{(r')}, \quad \Phi^{(r)}\Psi^{(s)}, \quad \Psi^{(s)}\Psi^{(s')}, \quad \Phi^{(r)}\Psi^{(s)}\Phi^{(r')} \quad \text{etc.} \tag{5.6}$$

[3]When dealing with real tensors, the position of the SO(8) indices $I, J, \ldots$ does not matter.

may be related to lower-order products. Consequently, one generates a list of group invariant tensors in the following way: In the beginning, the list only contains the selfdual and anti-selfdual tensors $\left\{\Phi^{(r)}_{IJKL}\right\}$ and $\left\{\Psi^{(s)}_{IJKL}\right\}$. Then, one iteratively constructs products of its elements, which may reduce to previously defined expressions in the list. If this is not the case, one defines them as new four-tensors (not necessarily selfdual or anti-selfdual) and adds them to the list of group invariants. This procedure stops when all products reproduce SO(8) objects already contained in the list. Exploiting all such identities should then enable us to compute $u_{IJ}{}^{KL}(x)$ and $v_{IJKL}(x)$ in a closed form. In other words, the sum in Eq. (5.4) becomes finite in terms of the elements in the list of group invariant tensors. This procedure will become more clear using the examples in Chapters 6 and 7.

The Ansätze may finally be brought into a form that is more adapted to the deformed S^7 geometry. Therefore, one uses the fact that the Killing forms introduced in Section 4.1 generate an orthogonal basis of selfdual and anti-selfdual SO(8) tensors (see Appendix A):

$$\begin{gathered}\text{selfdual}: \quad K_m{}^{[IJ}K^{m\,KL]}, \qquad K_{mn}{}^{[IJ}K^{n\,KL]}, \qquad K_m{}^{[IJ}K_n{}^{KL]} \\ \text{anti}-\text{selfdual}: \quad K_{[mn}{}^{[IJ}K_{p]}{}^{KL]}.\end{gathered} \tag{5.7}$$

In this basis, the selfdual and anti-selfdual four-forms $\Phi^{(r)}_{IJKL}$ and $\Psi^{(s)}_{IJKL}$ read

$$\begin{aligned}\Phi^{(r)}_{IJKL} &= \frac{1}{6}\xi^{(r)}K^{[IJ}_m K^{m\,KL]} - \frac{3}{2}\xi^{(r)}_{mn}K^{m\,[IJ}K^{n\,KL]} + \frac{1}{12}\xi^{(r)}_m K^{mn\,[IJ}K_n{}^{KL]}\,, \\ \Psi^{(s)}_{IJKL} &= \frac{1}{2}S^{(s)}_{mnp}\,K^{mn\,[IJ}K^{p\,KL]},\end{aligned} \tag{5.8}$$

where the corresponding components $\xi^{(r)}_m(y)$, $\xi^{(r)}_{mn}(y)$, $\xi^{(r)}(y)$, $S^{(s)}_{mnp}(y)$ are tensors defined on the round seven-sphere. Hence, its indices are raised and lowered with the background metric $\mathring{g}_{mn}$. Note that $S^{(s)}_{mnp}(y)$ is defined to be totally antisymmetric. The above relations may be inverted to obtain explicit expressions for the S^7 tensors. Therefore, one contracts them with the orthogonal (anti-)selfdual basis four-forms in Eq. (5.7) and uses Eqs. (A.6 – A.9). This yields

$$\xi^{(r)}_m = \frac{1}{16}\Phi^{(r)}_{IJKL}K_{mn}{}^{IJ}K^{n\,KL}\,, \quad \xi^{(r)}_{mn} = -\frac{1}{16}\Phi^{(r)}_{IJKL}K_m{}^{IJ}K_n{}^{KL}\,, \quad \xi^{(r)} = \mathring{g}^{mn}\xi^{(r)}_{mn} \tag{5.9}$$

for the scalars, and

$$S^{(s)}_{mnp} = \frac{1}{16}\Psi^{(s)}_{IJKL}K_{[mn}{}^{IJ}K_{p]}{}^{KL} \tag{5.10}$$

for the pseudo-scalars. The consequence of the performed steps is a huge simplification: All 11-dimensional fields can now be written solely in terms of the S^7 tensors and products of Killing forms. However, the SO(8) indices are fully contracted and as explained in Appendix A, all these contractions of Killing forms reduce to combinations of the invariant SO(7) tensors δ_m^n, $\mathring{g}_{mn}$ and $\mathring{\eta}_{m_1\cdots m_7}$, e.g.

$$K_m{}^{IJ}K_n{}^{IJ} = 8\mathring{g}_{mn}, \qquad K_m{}^{IJ}K_{np}{}^{IJ} = 0, \qquad K^{mn\,IJ}K_{pq}{}^{IJ} = 16\delta^{mn}_{pq}. \tag{5.11}$$

In other words, the 11-dimensional fields are finally written *only* in terms of the S^7 quantities (Eqs. (5.9, 5.10)):

$$\Big(g_{mn},\quad A_{mnp},\quad A_{m_1\cdots m_6},\quad \Delta \quad F_{mnpq},\quad \mathfrak{f}_{\rm FR},\Big)\,(x,y)$$
$$\Leftrightarrow$$
$$\Big(\Big\{\lambda^{(r)},\quad \mu^{(s)}\Big\}\,(x),\quad \Big\{\xi_m^{(r)},\quad \xi_{mn}^{(r)},\quad \xi^{(r)},\quad S_{mnp}^{(s)}\Big\}\,(y),\quad \Big\{\mathring{g}_{mn},\quad \mathring{\eta}_{m_1\cdots m_7},\quad \mathring{\zeta}^m\Big\}\,(y)\Big)\,.$$

The resulting expressions may still involve certain contractions between the S^7 quantities $\xi_m^{(r)}(y)$, $\xi_{mn}^{(r)}(y)$, $\xi^{(r)}(y)$, $S_{mnp}^{(s)}(y)$. Depending on the symmetry group, these contractions may further be simplified. Therefore, one inserts the explicit decomposition in Eq. (5.8) into all relations between the group invariant tensors $\Big\{\Phi^{(r)}_{IJKL}\Big\}$, $\Big\{\Psi^{(s)}_{IJKL}\Big\}$ (see Eq. (5.6)). This gives a complete list of identities between the S^7 tensors, which can be used to bring the resulting expressions for the 11-dimensional fields into a suitable form. Without these relations, it would not be possible to perform the consistency checks discussed in the next section.

Apart from these specific relations for the S^7 tensors in Eqs. (5.9, 5.10), there are some general relations that are valid for any underlying symmetry. First, the derivatives of the S^7 tensors can be computed using Eq. (4.11):

$$\begin{aligned}
\mathring{D}_m\xi^{(r)} &= 2m_7\xi_m^{(r)}, & \mathring{D}_m\xi_n^{(r)} &= 6m_7\,\xi_{mn}^{(r)} - 2m_7\,\xi^{(r)}\mathring{g}_{mn},\\
\mathring{D}_m\xi_{np}^{(r)} &= \frac{1}{3}m_7\Big(\mathring{g}_{np}\xi_m^{(r)} - \mathring{g}_{m(n}\xi_{p)}^{(r)}\Big), & \mathring{D}_m S_{npq}^{(s)} &= \frac{1}{6}m_7\mathring{\eta}_{mnpq}{}^{rst}S_{rst}^{(s)}.
\end{aligned} \tag{5.12}$$

Secondly, one finds the useful identities

$$\Phi^{(r)}_{IJKL}K_m{}^{KL} = -2\xi_{mn}^{(r)}K^{n\,IJ} - \frac{1}{3}\xi_n^{(r)}K_m{}^{n\,IJ}, \tag{5.13}$$

$$\Phi^{(r)}_{IJKL}K_{mn}{}^{KL} = \frac{2}{3}\xi^{(r)}_{[m}K_{n]}{}^{IJ} + \Big(\frac{2}{3}\xi^{(r)}\mathring{g}_{mp}\mathring{g}_{nq} - 4\mathring{g}_{p[m}\xi^{(r)}_{n]q}\Big)\,K^{pq\,IJ}, \tag{5.14}$$

$$\Psi^{(s)}_{IJKL}K_m{}^{KL} = S^{(s)}_{mnp}K^{np\,IJ}, \tag{5.15}$$

$$\Psi^{(s)}_{IJKL}K_{mn}{}^{KL} = 2S^{(s)}_{mnp}K^{p\,IJ} - \frac{1}{6}\mathring{\eta}_{mn}{}^{p_1\cdots p_5}S^{(s)}_{p_1p_2p_3}K_{p_4p_5}{}^{IJ}, \tag{5.16}$$

which can be proved using the explicit decomposition in Eq. (5.8) and the identities in Appendix A[4]. Eqs. (5.12 – 5.16) are valid for all r and s.

[4]Similar formulae may be found for all the SO(8) four-tensors that are contained in the list of group invariants. However, this must be done separately for the specific cases and will be discussed in the next two chapters.

5.2. The Consistency Checks of the Solution

This section describes two kinds of consistency checks for the obtained group invariant fields. These tests will be performed in the following two chapters for the G_2 and $SO(3)\times SO(3)$ invariant solutions of 11-dimensional supergravity. Within the discussed bosonic uplift of $N=8$ supergravity to 11 dimensions, all fermions are consistently set to zero.

The first test verifies the duality relation in Eq. (3.12). Therefore, setting all fermions to zero, one considers the seven internal components, i.e.

$$\frac{i}{4!\cdot 7!}\epsilon_{m_1\cdots m_7\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma} = \mathring{D}_{[m_1}A_{m_2\cdots m_7]} + \frac{1}{4!\sqrt{2}}A_{[m_1\cdots m_3}F_{m_4\cdots m_7]}.$$

With the particular decomposition of the ϵ tensor,

$$\epsilon_{m_1\cdots m_7\mu\nu\rho\sigma} = \Delta^{-1}\mathring{\eta}_{m_1\cdots m_7}\mathring{\eta}_{\mu\nu\rho\sigma}, \tag{5.17}$$

one contracts the above duality relation with $\mathring{\eta}^{m_1\cdots m_7}$ and finds

$$\frac{i\Delta^{-1}}{4!}\mathring{\eta}_{\mu\nu\rho\sigma}F^{\mu\nu\rho\sigma} = \mathring{\eta}^{m_1\cdots m_7}\left(\mathring{D}_{m_1}A_{m_2\cdots m_7} + \frac{1}{4!\sqrt{2}}A_{m_1\cdots m_3}F_{m_4\cdots m_7}\right).$$

Furthermore, using Eqs. (3.15, 4.58) on the lhs and Eq. (4.39) on the rhs finally results in

$$\mathfrak{f}_{\mathrm{FR}} = -\Delta^{-3}\left(\frac{1}{6}\mathring{D}_n\left(\Delta A^n\right) + \frac{1}{\sqrt{2}\cdot 4!}\mathring{\eta}^{m_1\cdots m_7}A_{m_1m_2m_3}F_{m_4m_5m_6m_7}\right). \tag{5.18}$$

This relation must hold off-shell, i.e. it does not require the equations of motion to be satisfied. Hence, Eq. (5.18) represents a very non-trivial consistency check for the uplift Ansätze of the internal form potentials (Eqs. (4.39, 4.49)), the field-strength (Eq. (4.56)) and the Freund-Rubin term (Eq. (4.64)).

The second test is the verification of the equations of motion for the Freund-Rubin compactification[5]. In this case, Eqs. (3.8, 3.9) simplify to

$$R_{\mu\nu} = \left(\frac{\Delta^{-1}}{72}F_{mnpq}F^{mnpq} + \frac{2}{3}f_{\mathrm{FR}}^2\Delta^3\right)\Delta g_{\mu\nu}, \tag{5.19}$$

$$R_{mn} = \left(\frac{\Delta^{-1}}{72}F_{mnpq}F^{mnpq} - \frac{1}{3}f_{\mathrm{FR}}^2\Delta^3\right)\Delta g_{mn} - \frac{1}{6}F_{mpqr}F_n{}^{pqr}, \tag{5.20}$$

$$\mathring{D}_q\left(\Delta^{-1}F^{mnpq}\right) = \frac{\sqrt{2}}{24}\mathfrak{f}_{\mathrm{FR}}\mathring{\eta}^{mnpqrst}F_{qrst}. \tag{5.21}$$

[5] The spacetime is maximally symmetric and all fields are evaluated at the group invariant stationary point of the scalar potential.

In order to verify these three equations, one must compute the following scalar fields:

$$g_{mn}, \quad A_{mnp}, \quad \Delta, \quad F_{mnpq}, \quad F^{mnpq}, \quad \mathfrak{f}_{\mathrm{FR}}, \quad R_{\mu\nu}, \quad R_{mn}.$$

In particular, these can all be found using the scalar uplift Ansätze in Chapter 4, except for the components of the Ricci tensor.

Therefore, the rest of this section now explicitly shows, how one computes the components $R_{\mu\nu}$ and R_{mn} on the lhs of Eqs. (5.19, 5.20). From Eq. (3.15), one has

$$g_{\mu\nu}(x,y) = \Delta^{-1}(y)\mathring{g}_{\mu\nu}(x), \qquad g^{\mu\nu}(x,y) = \Delta(y)\mathring{g}^{\mu\nu}(x), \tag{5.22}$$

and the various Christoffel symbols read

$$\begin{gathered}\Gamma^{\rho}_{\mu\nu} = \mathring{\Gamma}^{\rho}_{\mu\nu}, \qquad \Gamma^{\rho}_{mn} = \Gamma^{p}_{m\nu} = 0, \qquad \Gamma^{p}_{\mu\nu} = \frac{1}{2} g_{\mu\nu} g^{pq} \mathring{D}_q \log\Delta, \\ \Gamma^{\rho}_{\mu n} = -\frac{1}{2}\delta^{\rho}_{\mu}\mathring{D}_n \log\Delta, \qquad \Gamma^{p}_{mn} = \hat{\Gamma}^{p}_{mn} + \mathring{\Gamma}^{p}_{mn}.\end{gathered} \tag{5.23}$$

Here, the connection components with a circle on top denote the background Christoffel symbols (of AdS_4 and the round seven-sphere). Moreover,

$$\hat{\Gamma}^{p}_{mn} = \frac{1}{2} g^{pq}\left(\mathring{D}_m g_{nq} + \mathring{D}_n g_{mq} - \mathring{D}_q g_{mn}\right) \tag{5.24}$$

denotes a convenient *tensor*.

The relevant components of the eleven-dimensional Riemann tensor are

$$\begin{aligned} R^{\mu}{}_{\nu\rho\sigma} &= -\partial_\rho \Gamma^{\mu}_{\sigma\nu} + \partial_\sigma \Gamma^{\mu}_{\rho\nu} - \Gamma^{\mu}_{\rho M}\Gamma^{M}_{\sigma\nu} + \Gamma^{\mu}_{\sigma N}\Gamma^{N}_{\rho\nu} \\ &= \mathring{R}^{\mu}{}_{\nu\rho\sigma} + \frac{1}{2}\delta_{[\rho}{}^{\mu} g_{\sigma]\nu} g^{pq} \mathring{D}_p \log\Delta\, \mathring{D}_q \log\Delta, \end{aligned} \tag{5.25}$$

$$\begin{aligned} R^{\mu}{}_{m\nu n} &= -\partial_\nu \Gamma^{\mu}_{nm} + \partial_n \Gamma^{\mu}_{\nu m} - \Gamma^{\mu}_{\nu p}\Gamma^{p}_{nm} + \Gamma^{\mu}_{n\rho}\Gamma^{\rho}_{\nu m} \\ &= \frac{1}{2}\delta^{\mu}_{\nu}\left(-\mathring{D}_m\mathring{D}_n \log\Delta + \hat{\Gamma}^{p}_{mn}\mathring{D}_p \log\Delta + \frac{1}{2}\mathring{D}_m \log\Delta\, \mathring{D}_n \log\Delta)\right), \end{aligned} \tag{5.26}$$

$$R^{m}{}_{\mu n\nu} = g^{mp} g_{\mu\rho} R^{\rho}{}_{p\nu n}, \tag{5.27}$$

$$R^{m}{}_{npq} = \mathring{R}^{m}{}_{npq} - \mathring{D}_p \hat{\Gamma}^{m}_{qn} + \mathring{D}_q \hat{\Gamma}^{m}_{pn} - \hat{\Gamma}^{m}_{pr}\hat{\Gamma}^{r}_{qn} + \hat{\Gamma}^{m}_{qr}\hat{\Gamma}^{r}_{pn}. \tag{5.28}$$

Here, $\mathring{R}^{\mu}{}_{\nu\rho\sigma}$ and $\mathring{R}^{m}{}_{npq}$ denote the Riemann tensors of the background AdS_4 space and the round seven-sphere, respectively. The associated Ricci tensors in our conventions are

$$\mathring{R}_{\mu\nu} = 3m_4^2 \mathring{g}_{\mu\nu}, \qquad \mathring{R}_{mn} = -6m_7^2 \mathring{g}_{mn}, \tag{5.29}$$

where m_4 denotes the inverse AdS_4 radius. It is related to the S^7 radius via the scalar potential at the stationary point $V_\star$ [51],

$$m_4^2 = -\frac{2V_\star}{3g^2} m_7^2. \tag{5.30}$$

It is now straightforward to obtain the expressions for the relevant components of the Ricci tensor. In particular,

$$R_{\mu\nu} = \left(3m_4^2 - \frac{1}{2}\mathring{D}_m\left(\Delta^{-1}g^{mn}\mathring{D}_n\log\Delta\right)\right)\Delta g_{\mu\nu}, \tag{5.31}$$

$$\begin{aligned} R_{mn} = &-6m_7^2\mathring{g}_{mn} + \tilde{\Gamma}^q_{mp}\tilde{\Gamma}^p_{qn} - \mathring{D}_p\tilde{\Gamma}^p_{mn} \\ &- \frac{9}{4}\mathring{D}_m\log\Delta\,\mathring{D}_n\log\Delta - \frac{1}{2}\mathring{D}_p\left(\Delta^{-1}g^{pq}\mathring{D}_q\log\Delta\right)\Delta g_{mn}, \end{aligned} \tag{5.32}$$

where one defines the convenient tensor

$$\tilde{\Gamma}^p_{mn} = \frac{1}{2}\Delta^{-1}g^{pq}\left(\mathring{D}_m\Delta g_{nq} + \mathring{D}_n\Delta g_{mq} - \mathring{D}_q\Delta g_{mn}\right). \tag{5.33}$$

The following two Chapters construct the G_2 and SO(3)×SO(3) invariant solutions of 11-dimensional supergravity. In particular, Eqs. (5.18 – 5.21) are explicitly verified for these solutions.

6. G_2 Invariant Supergravity

This chapter repeats the construction of the G_2 invariant solution of 11-dimensional supergravity. The uplift Ansatz for the inverse internal metric was already known in the early 1980s. In particular, the authors in Ref. [43] derived the explicit expression for $\Delta^{-1}g^{mn}$ within the G_2 invariant solution of 11-dimensional supergravity. The obtained matrix can be inverted to derive $\Delta g_{mn}(x,y)$ and the warp factor can be removed by explicitly taking the determinant. The three-form Ansatz was found 20 years later [45] and was verified for the G_2 invariant solution of 11-dimensional supergravity [60]. In particular, the authors of the latter paper found the explicit expression for the internal four-form field-strength using Eq. (4.50) and verified that the Freund-Rubin solution satisfies the simplified equations of motions in Eqs. (5.19 – 5.21). Finally, the explicit uplift Ansätze for the internal metric $g_{mn}(x,y)$ and field-strength $F_{mnpq}(x,y)$ as well as the Freund-Rubin term $\mathfrak{f}_{\mathrm{FR}}(x,y)$ (Eqs. (4.45, 4.56, 4.64)) have been explicitly checked for the G_2 invariant solution of 11-dimensional supergravity [48, 49].

This chapter summarizes these results in three steps: The first part gives the decomposition of the scalar fields ${u_{ij}}^{IJ}(x)$ and $v_{ij\,IJ}(x)$ into G_2 invariant tensors. It also introduces the explicit S^7 quantities (Eqs. (5.9, 5.10)) and gives the corresponding identities. Section 6.2 then constructs the 11-dimensional fields using the explicit uplift Ansätze derived in Chapter 4. Finally, Section 6.3 checks the consistency for the derived Freund-Rubin solution as explained in Section 5.2.

6.1. G_2 Invariant Tensors and Corresponding S^7 Quantities

Let us now explicitly perform the steps that were discussed in Section 5.1. This includes to give the explicit decomposition of the ${u_{ij}}^{IJ}(x)$ and $v_{ij\,IJ}(x)$ tensors into G_2 invariant objects and to find all corresponding identities that are required to derive the G_2 invariant solution of 11-dimensional supergravity.

First, there is only one selfdual and one anti-selfdual G_2 invariant tensor, i.e.

$$\Phi^{(0)}_{IJKL} = C^{+}_{IJKL}, \qquad \Psi^{(0)}_{IJKL} = C^{-}_{IJKL}. \tag{6.1}$$

Together with a reparametrization of the scalar coordinates $\lambda^{(0)}(x)$ and $\mu^{(0)}(x)$ into a scalar field $\lambda(x)$ and a rotation angle $\alpha(x)$,

$$\lambda^{(0)}(x) = \frac{\lambda(x)}{2}\cos\alpha(x), \qquad \mu^{(0)}(x) = \frac{\lambda(x)}{2}\sin\alpha(x),$$

one finds the explicit expression for the scalar vacuum expectation value $\phi_{IJKL}(x)$. In particular, Eq. (5.5) reduces to

$$\phi_{IJKL} = \frac{\lambda}{2}\left(C^+_{IJKL}\cos\alpha + iC^-_{IJKL}\sin\alpha\right). \tag{6.2}$$

Secondly, one constructs a list of G_2 invariant four-tensors, which are not necessarily selfdual or anti-selfdual. Therefore, the only relations for products of the $C^\pm_{IJKL}$ tensors are [53]

$$\left(C^\pm\right)^2_{IJKL} = C^\pm_{IJMN}C^\pm_{MNKL} = 12\delta^{IJ}_{KL} \pm 4C^\pm_{IJKL}. \tag{6.3}$$

Since there is no such identity for the products C^+C^- and C^-C^+, they must define new G_2 invariants. Hence, our list of G_2 invariant four-tensors reads

$$\delta^{IJ}_{KL}, \qquad C^\pm_{IJKL}, \qquad D^\pm_{IJKL} = \frac{1}{2}\left(C^+_{IJMN}C^-_{MNKL} \pm C^-_{IJMN}C^+_{MNKL}\right).$$

In principle, this list may be extended by cubic terms like $C^+C^-C^+$ etc., but it turns out that no further definitions are required for our purposes.

One now relates the four-dimensional scalars ${u_{ij}}^{IJ}(x)$ and $v_{ij\,IJ}(x)$ to the above G_2 invariants. In particular, Eq. (5.4) simplifies using the explicit form of the vacuum expectation value in Eq. (6.2) and the contraction identities in Eq. (6.3):

$${u_{IJ}}^{KL} = p^3\delta^{IJ}_{KL} + \frac{1}{2}pq^2\cos^2\alpha C^+_{IJKL} - \frac{1}{2}pq^2\sin^2\alpha C^-_{IJKL} - \frac{i}{8}pq^2\sin 2\alpha D^-_{IJKL}, \tag{6.4}$$

$$\begin{aligned} v_{IJKL} = q^3(\cos^3\alpha - i\sin^3\alpha)\delta^{IJ}_{KL} + \frac{1}{2}p^2q\cos\alpha C^+_{IJKL} \\ + \frac{i}{2}p^2q\sin\alpha C^-_{IJKL} - \frac{1}{8}q^3\sin 2\alpha(\sin\alpha - i\cos\alpha)D^+_{IJKL}, \end{aligned} \tag{6.5}$$

where $p(x) = \cosh\lambda(x)$ and $q(x) = \sinh\lambda(x)$.

The explicit G_2 invariant S^7 quantities are abbreviated as[1]

$$\xi^{(0)} = \xi, \qquad \xi^{(0)}_m = \xi_m, \qquad \xi^{(0)}_{mn} = \xi_{mn}, \qquad S^{(0)}_{mnp} = S_{mnp},$$

[1] In Ref. [60], S_{mnp} was denoted by $\mathring{S}_{mnp}$.

and Eqs. (5.8 – 5.10) read

$$\begin{aligned} C^{+}_{IJKL} &= \frac{\xi}{6} K_m{}^{[IJ} K^{m\,KL]} - \frac{3}{2} \xi^{mn} K_m{}^{[IJ} K_n{}^{KL]} + \frac{1}{12} \xi^m K_{mn}{}^{[IJ} K^{n\,KL]}, \\ C^{-}_{IJKL} &= \frac{1}{2} S^{mnp} K_{mn}{}^{[IJ} K_p{}^{KL]}, \end{aligned} \tag{6.6}$$

$$\begin{aligned} \xi_m = \frac{1}{16} C^{+}_{IJKL} K^{IJ}_{mn} K^{n\,KL}\ , \quad \xi_{mn} &= -\frac{1}{16} C^{+}_{IJKL} K^{IJ}_m K^{KL}_n\ , \quad \xi = \mathring{g}^{mn} \xi_{mn}, \\ S_{mnp} &= \frac{1}{16} C^{-}_{IJKL} K_{[mn}{}^{IJ} K_{p]}{}^{KL}. \end{aligned} \tag{6.7}$$

Furthermore, inserting Eq. (6.6) into the contraction relations in Eq. (6.3) yields the useful specific identities [60]

$$\xi^{mn} \mathring{g}_{mn} = \xi, \quad \xi_m \xi_n = (9 - \xi^2) \mathring{g}_{mn} - 6(3 - \xi) \xi_{mn}, \quad \xi_m \xi^m = (21 + \xi)(3 - \xi), \tag{6.8}$$

$$\begin{aligned} S^{m[np} S^{qr]s} = \frac{1}{6} \mathring{\eta}^{npqr(m}{}_{tu} S^{s)tu}, \quad S^{[mnp} S^{q]rs} &= \frac{1}{4} \mathring{\eta}^{mnpq[r}{}_{tu} S^{s]tu}, \\ S^{mnr} S_{pqr} = 2\delta^{mn}_{pq} &+ \frac{1}{6} \mathring{\eta}^{mn}{}_{pqrst} S^{rst}. \end{aligned} \tag{6.9}$$

Finally, the general identities in Eqs. (5.12 – 5.16) translate to

$$\begin{aligned} \mathring{D}_m \xi &= 2m_7 \xi_m, & \mathring{D}_m \xi_n &= 6m_7\, \xi_{mn} - 2m_7\, \xi \mathring{g}_{mn}, \\ \mathring{D}_m \xi_{np} &= \frac{1}{3} m_7 \Big(\mathring{g}_{np} \xi_m - \mathring{g}_{m(n} \xi_{p)} \Big), & \mathring{D}_m S_{npq} &= \frac{1}{6} m_7 \mathring{\eta}_{mnpq}{}^{rst} S_{rst}, \end{aligned} \tag{6.10}$$

and

$$C_{+}^{IJKL} K_m{}^{KL} = -2\xi_{mn} K^{n\,IJ} - \frac{1}{3} \xi^n K_{mn}{}^{IJ}, \tag{6.11}$$

$$C_{+}^{IJKL} K_{mn}{}^{KL} = \frac{2}{3} \xi_{[m} K_{n]}{}^{IJ} + \left(\frac{2}{3} \xi \mathring{g}_{mp} \mathring{g}_{nq} - 4 \mathring{g}_{p[m} \xi_{n]q} \right) K^{pq\,IJ}, \tag{6.12}$$

$$C_{-}^{IJKL} K_m{}^{KL} = S_{mnp} K^{np\,IJ}, \tag{6.13}$$

$$C_{-}^{IJKL} K_{mn}{}^{KL} = 2S_{mnp} K^{p\,IJ} - \frac{1}{6} \mathring{\eta}_{mn}{}^{p_1 \cdots p_5} S_{p_1 p_2 p_3} K_{p_4 p_5}{}^{IJ}. \tag{6.14}$$

It is also convenient to compute the following single contractions of the specific G_2

invariant $D^{\pm}$ tensors with the Killing forms:

$$D_{+}^{IJKL}K_m{}^{KL} = \left(\frac{\xi}{3}S_{mnp} - \xi_m{}^q S_{npq} - 2S_{mnq}\xi_p{}^q + \frac{1}{36}\mathring{\eta}_{mnpqrst}\xi^q S^{rst}\right) K^{np\,IJ}, \tag{6.15}$$

$$\begin{aligned} D_{+}^{IJKL}K_{mn}{}^{KL} = &\left(\frac{2}{3}\xi S_{mnp} - 4\xi_{[m}{}^q S_{n]pq} - 2S_{mnq}\xi_p{}^q + \frac{1}{18}\mathring{\eta}_{mnpqrst}\xi^q S^{rst}\right) K^{p\,IJ} \\ &+ \frac{1}{3}\Big(\xi_{[m}S_{n]pq} - S_{mnp}\xi_q - \frac{\xi}{3}\mathring{\eta}_{mnpqrst}S^{rst} \\ &\qquad + \mathring{\eta}_{mnprstu}\xi_q{}^r S^{stu} + \mathring{\eta}_{[m|pqrstu}\xi_{n]}{}^r S^{stu}\Big) K^{pq\,IJ}, \end{aligned} \tag{6.16}$$

$$\begin{aligned} D_{-}^{IJKL}K_m{}^{KL} = &\frac{2}{3}S_{mnp}\xi^n K^{p\,IJ} \\ &+ \left(\frac{\xi}{3}S_{mnp} + \xi_m{}^q S_{npq} - 2S_{mnq}\xi_p{}^q - \frac{1}{36}\mathring{\eta}_{mnpqrst}\xi^q S^{rst}\right) K^{np\,IJ}, \end{aligned} \tag{6.17}$$

$$\begin{aligned} D_{-}^{IJKL}K_{mn}{}^{KL} = &\left(-\frac{2}{3}\xi S_{mnp} + 4\xi_{[m}{}^q S_{n]pq} - 2S_{mnq}\xi_p{}^q + \frac{1}{18}\mathring{\eta}_{mnpqrst}\xi^q S^{rst}\right) K^{p\,IJ} \\ &+ \frac{1}{3}\left(-\xi_{[m}S_{n]pq} - S_{mnp}\xi_q + \mathring{\eta}_{mnprstu}\xi_q{}^r S^{stu} - \mathring{\eta}_{[m|pqrstu}\xi_{n]}{}^r S^{stu}\right) K^{pq\,IJ}. \end{aligned} \tag{6.18}$$

These identities can be proved using the explicit definition of the $D^{\pm}$ tensors and the relations in Eqs. (6.11 – 6.14).

6.2. Constructing the G_2 Invariant Supergravity Solution

This section gives the explicit calculations to derive all 11-dimensional scalar fields in terms of the G_2 invariant S^7 tensors in Eq. (6.7). These constitute the complete Freund-Rubin solution with flux. All computations are performed with the computer algebra program `FORM` [61, 62]. Therefore, the main steps to derive the solution are described in detail but only some intermediate results are given.

Let us start with the uplift Ansatz for the inverse metric in Eq. (4.30). With the explicit decomposition of the $u_{ij}{}^{IJ}(x)$ and $v_{ij\,IJ}(x)$ tensors into the G_2 invariants in Eqs. (6.4, 6.5) and the contraction identities in Eqs. (6.11 – 6.18), one finds that

$$K^{m\,KL}u_{IJ}{}^{KL} = u_1^{mn}K_n{}^{IJ} + u_1^{mnp}K_{np}{}^{IJ}, \quad K^{m\,KL}v_{IJ\,KL} = v_1^{mn}K_n{}^{IJ} + v_1^{mnp}K_{np}{}^{IJ}. \tag{6.19}$$

Here, the explicit components are

$$
\begin{aligned}
u_1^{mn} &= p^3\mathring{g}^{mn} - pq^2\cos^2\alpha\,\xi^{mn} + \frac{i}{12}pq^2\sin 2\alpha\, S^{mnp}\xi_p, \\
u_1^{mnp} &= -\frac{1}{6}pq^2\cos^2\alpha\,\mathring{g}^{m[n}\xi^{p]} - \frac{1}{2}pq^2\sin^2\alpha S^{mnp} \\
&\quad + \frac{i}{8}pq^2\sin 2\alpha\left(2S^{m[n}{}_q\xi^{p]q} - \xi^m{}_q S^{npq} - \frac{\xi}{3}S^{mnp} + \frac{1}{36}\mathring{\eta}^{mnpqrst}\xi_q S_{rst}\right), \\
v_1^{mn} &= q^3(\cos^3\alpha - i\sin^3\alpha)\mathring{g}^{mn} - p^2q\cos\alpha\,\xi^{mn}, \\
v_1^{mnp} &= -\frac{1}{6}p^2q\cos\alpha\,\mathring{g}^{m[n}\xi^{p]} + \frac{i}{2}p^2q\sin\alpha S^{mnp} + \frac{1}{8}q^3\sin 2\alpha \\
&\quad \times(\sin\alpha - i\cos\alpha)\left(2S^{m[n}{}_q\xi^{p]q} + \xi^m{}_q S^{npq} - \frac{\xi}{3}S^{mnp} - \frac{1}{36}\mathring{\eta}^{mnpqrst}\xi_q S_{rst}\right).
\end{aligned}
$$

Using these relations together with Eqs. (5.11, 6.8, 6.9), the uplift Ansatz for the inverse metric (Eq. (4.30)) finally reduces to

$$
\Delta^{-1}g^{mn} = \left(c^3 + v^3s^3\right)\mathring{g}^{mn} - cvs(c+vs)\xi^{mn}, \tag{6.20}
$$

where $c(x) = \cosh 2\lambda(x)$, $s(x) = \sinh 2\lambda(x)$, and $v(x) = \cos\alpha(x)$.

Secondly, $\Delta^{-1}A_{mn}{}^p$ may be computed analogously using the uplift Ansatz in Eq. (4.31). Therefore, one needs to calculate

$$
\begin{aligned}
K_{mn}{}^{KL}u_{IJ}{}^{KL} &= u_{2\,mnp}K^{p\,IJ} + u_{2\,mn}{}^{pq}K_{pq}{}^{IJ}, \\
K_{mn}{}^{KL}v_{IJ\,KL} &= v_{2\,mnp}K^{p\,IJ} + v_{2\,mn}{}^{pq}K_{pq}{}^{IJ}.
\end{aligned} \tag{6.21}
$$

The explicit components are

$$
\begin{aligned}
u_{2\,mnp} &= \frac{1}{3} pq^2 \cos^2\alpha\, \xi_{[m}\mathring{g}_{n]p} - pq^2 \sin^2\alpha S_{mnp} \\
&\quad + \frac{i}{4} pq^2 \sin 2\alpha \left(S_{mn}{}^{q}\xi_{pq} - 2\xi_{[m}{}^{q}S_{n]pq} + \frac{\xi}{3} S_{mnp} - \frac{1}{36}\mathring{\eta}_{mnpqrst}\xi^{q}S^{rst} \right), \\
u_{2\,mn}{}^{pq} &= \left(\frac{\xi}{3} pq^2 \cos^2\alpha + p^3 \right) \delta^{pq}_{mn} - 2pq^2\cos^2\alpha\, \xi_{[m}{}^{[p}\delta_{n]}{}^{q]} + \frac{1}{12} pq^2 \sin^2\alpha\, \mathring{\eta}_{mn}{}^{pqrst}S_{rst} \\
&\quad + \frac{i}{24} pq^2 \sin 2\alpha \left(\xi_{[m}S_{n]}{}^{pq} - \xi^{[p}S^{q]}{}_{mn} - \xi_{r[m}\mathring{\eta}_{n]}{}^{pqrstu}S_{stu} - \xi_{r}{}^{[q}\mathring{\eta}_{mn}{}^{p]rstu}S_{stu} \right), \\
v_{2\,mnp} &= \frac{1}{3} p^2 q \cos\alpha\, \xi_{[m}\mathring{g}_{n]p} + ip^2 q \sin\alpha S_{mnp} + \frac{1}{4} q^3 \sin 2\alpha (\sin\alpha - i\cos\alpha) \\
&\qquad \times \left(S_{mn}{}^{q}\xi_{pq} + 2\xi_{[m}{}^{q}S_{n]pq} - \frac{\xi}{3} S_{mnp} - \frac{1}{36}\mathring{\eta}_{mnpqrst}\xi^{q}S^{rst} \right), \\
v_{2\,mn}{}^{pq} &= \left(\frac{\xi}{3} p^2 q \cos\alpha + q^3(\cos^3\alpha - i\sin^3\alpha) \right) \delta^{pq}_{mn} - 2p^2 q \cos\alpha\, \xi_{[m}{}^{[p}\delta_{n]}{}^{q]} \\
&\quad + \left(-\frac{i}{12} p^2 q \sin\alpha + \frac{\xi}{72} q^3 \sin 2\alpha (\sin\alpha - i\cos\alpha) \right) \mathring{\eta}_{mn}{}^{pqrst}S_{rst} \\
&\quad - \frac{1}{12} q^3 \sin 2\alpha (\sin\alpha - i\cos\alpha) \mathring{g}^{pr}\mathring{g}^{qs} \left(\xi_{[m}S_{nrs]} + \mathring{\eta}_{[mnr}{}^{tuvw}\xi_{s]t}S_{uvw} \right).
\end{aligned}
$$

Again, these relations together with Eqs. (5.11, 6.8, 6.9) simplify the uplift Ansatz in Eq. (4.31) to

$$
\begin{aligned}
\Delta^{-1} A_{mn}{}^{p} = \frac{\sqrt{2} v^2 s^2 \tan\alpha}{24} \Bigg[& 2(c - vs)\xi_{q[m}S_{n]}{}^{pq} - (c + vs) S_{mnq}\xi^{pq} \\
& + \frac{1}{36}(c + vs)\mathring{\eta}_{mn}{}^{pqrst}\xi_{q}S_{rst} - \left(\frac{c - vs}{3}\xi - 2\frac{c^2}{vs} \right) S_{mn}{}^{p} \Bigg].
\end{aligned} \tag{6.22}
$$

Both, the inverse metric (Eq. (6.20)) and the internal three-form with mixed index structure (Eq. (6.22)) agree with the formulae obtained in Ref. [60]. In that paper, the metric Δg_{mn} was found by explicitly inverting the expression in Eq. (6.20) and the warp factor was removed by taking the determinant. Finally, the full three-form potential A_{mnp} could be computed by lowering the remaining upper index of the expression in Eq. (6.22) . This procedure might be simple for the G$_2$ invariant solution but will be rather complicated for the SO(3)×SO(3) case. Therefore, this thesis obtains the metric via the direct uplift Ansatz in Eq. (4.45) [49].

The metric Ansatz in Eq. (4.45) requires the tensors $\mathcal{A}_{m\,ijkl}$ and $\mathcal{B}_{m\,ijkl}$ (or better: $\mathcal{A}_{m\,IJKL}$, $\mathcal{B}_{m\,IJKL}$) defined in Eqs. (4.43, 4.44). In particular, using Eqs. (A.16, A.17) yields

$$\mathcal{A}_{mIJKL} = \frac{1}{8}\left(K_{mn}{}^{MN}K^{nPQ} + K_{mn}{}^{PQ}K^{nMN}\right)\left(u_{IJ}{}^{MN}u_{KL}{}^{PQ} - v_{IJ\,MN}v_{KL\,PQ}\right), \quad (6.23)$$

$$\mathcal{B}_{mIJKL} = \frac{1}{8}\left(K_{mn}{}^{PQ}K^{nMN} - K_{mn}{}^{MN}K^{nPQ}\right)\left(u_{IJ}{}^{MN}v_{KL\,PQ} - v_{IJ\,MN}u_{KL}{}^{PQ}\right). \quad (6.24)$$

Although this does not seem to be a simplification, one may now perform the same steps as before: One uses the explicit decomposition of the scalar fields $u_{ij}{}^{IJ}(x)$ and $v_{ij\,IJ}(x)$ in Eqs. (6.4, 6.5) and the contraction identities in Eqs. (6.11 – 6.18). Since the result is antisymmetric in the SO(8) indices, one may write the $\mathcal{A}_{m\,IJKL}$ and $\mathcal{B}_{m\,IJKL}$ tensors in the basis provided by the Killing forms (Eq. (5.7)),

$$\begin{aligned}\mathcal{A}_{mIJKL} = {}& \frac{1}{6}\,a_m K_n{}^{[IJ}K^{n\,KL]} - \frac{3}{2}\,a_m{}^{np}K_n{}^{[IJ}K_p{}^{KL]} \\ & + \frac{1}{12}\,a_m{}^n K_{np}{}^{[IJ}K^{p\,KL]} + \frac{1}{2}\,a_m{}^{npq}K_{[np}{}^{[IJ}K_{q]}{}^{KL]}, \end{aligned} \quad (6.25)$$

$$\begin{aligned}\mathcal{B}_{mIJKL} = {}& \frac{1}{6}\,b_m K_n{}^{[IJ}K^{n\,KL]} - \frac{3}{2}\,b_m{}^{np}K_n{}^{[IJ}K_p{}^{KL]} \\ & + \frac{1}{12}\,b_m{}^n K_{np}{}^{[IJ}K^{p\,KL]} + \frac{1}{2}\,b_m{}^{npq}K_{[np}{}^{[IJ}K_{q]}{}^{KL]}. \end{aligned} \quad (6.26)$$

The components can be found by contracting these relations with the basis (anti-)selfdual forms in Eq. (5.7). In particular, using Eqs. (A.6 – A.9) in Appendix A yields

$$\begin{aligned} a_m &= a_m{}^{np}\mathring{g}_{np}, & b_m &= b_m{}^{np}\mathring{g}_{np}, \\ a_m{}^n &= \frac{1}{16}\mathcal{A}_{m\,IJKL}K^{np[IJ}K_p{}^{KL]}, & b_m{}^n &= \frac{1}{16}\mathcal{B}_{m\,IJKL}K^{np[IJ}K_p{}^{KL]}, \\ a_m{}^{np} &= -\frac{1}{16}\mathcal{A}_{m\,IJKL}K^{n\,[IJ}K^{p\,KL]}, & b_m{}^{np} &= -\frac{1}{16}\mathcal{B}_{m\,IJKL}K^{n\,[IJ}K^{p\,KL]}, \\ a_m{}^{npq} &= \frac{1}{16}\mathcal{A}_{m\,IJKL}K^{np[IJ}K^{q\,KL]}, & b_m{}^{npq} &= \frac{1}{16}\mathcal{B}_{m\,IJKL}K^{np[IJ}K^{q\,KL]}, \end{aligned}$$

where

$$\begin{aligned}
a_m &= -\frac{v^2 s^2}{24}(\xi+9)\xi_m,\\
a_m{}^n &= \left(\frac{3-\xi}{4}v^2 s^2 + 3v^2 + 3c - 3cv^2\right)\delta_m{}^n + \frac{v^2 s^2}{24}\left(\frac{3+\xi}{3-\xi}\right)\xi_m\xi^n,\\
a_m{}^{np} &= -\frac{v^2 s^2}{144}\left(12\delta_m{}^{(n}\xi^{p)} + (9+\xi)\xi_m\mathring{g}^{np} - \frac{1}{3-\xi}\xi_m\xi^n\xi^p\right),\\
a_m{}^{npq} &= \frac{i\tan\alpha}{48}v^2(1-c)\Big(\xi_m S^{npq} + 3cS_m{}^{[np}\xi^{q]} - 3(1+c)\delta_m{}^{[n}S^{pq]r}\xi_r + c\xi_{rm}\mathring{\eta}^{npqrstu}S_{stu}\\
&\qquad + 3\xi_r{}^{[n}\mathring{\eta}_m{}^{pq]rstu}S_{stu} + \frac{\xi}{3}(c-1)\mathring{\eta}_m{}^{npqrst}S_{rst}\Big),\\
b_m &= -\frac{vs}{4}(1-v^2+c+cv^2)\xi_m,\\
b_m{}^n &= \frac{vs}{4}\left(1-v^2+c+cv^2\right)(\xi\delta_m{}^n - 3\xi_m{}^n) + \frac{vs}{8}\left(1-v^2-c+cv^2\right)S_m{}^{np}\xi_p,\\
b_m{}^{np} &= \frac{vs}{24}(1-v^2+c+cv^2)\left(\delta_m{}^{(n}\xi^{p)} - \xi_m\mathring{g}^{np}\right) + \frac{vs}{4}(1-v^2-c+cv^2)S_{qm}{}^{(n}\xi^{p)q},\\
b_m{}^{npq} &= \frac{i\tan\alpha}{48}v^3 s(1-c)\Big(-3S_m{}^{[np}\xi^{q]} + \xi_{rm}\mathring{\eta}^{npqrstu}S_{stu}\\
&\qquad + 3\delta_m{}^{[n}S^{pq]r}\xi_r - \left(1+\frac{2c}{v^2(1-c)}\right)\mathring{\eta}_m{}^{npqrst}S_{rst}\Big).
\end{aligned}$$

Finally, one computes the metric via the uplift Ansatz in Eq. (4.45) using the explicit expressions for the $\mathcal{A}_{m\,IJKL}$ and $\mathcal{B}_{m\,IJKL}$ tensors in Eqs. (6.25, 6.26). The contractions of the SO(8) indices requires Eqs. (A.6 – A.9) and the SO(7) indices $m, n, \ldots$ may be contracted using the identities in Eqs. (6.8, 6.9). The resulting expression is

$$\Delta^{-2}g_{mn} = b_0\Big[(b_0 + 3cvs)\,\mathring{g}_{mn} + cvs\,\xi_{mn}\Big], \tag{6.27}$$

where

$$b_0(\xi) = c^2 + v^2 s^2 - \frac{9+\xi}{6}\,cvs. \tag{6.28}$$

Combining the explicit expressions for the metric and its inverse in Eqs. (6.27, 6.20) and using the identities in Eqs. (6.8, 6.9) gives

$$\Delta^{-2}g_{mp}\Delta^{-1}g^{pn} = b_0^2\,(c+vs)^3\delta_m^n.$$

This is exactly the combination of the metric and its inverse that defines the warp factor in Eq. (4.46). In particular, g_{mn} and g^{mn} are inverse for

$$\Delta^{-3} = b_0^2\,(c+vs)^3. \tag{6.29}$$

The metric and the warp factor in Eqs. (6.27, 6.29) coincide with the results of Ref. [60]. In particular, the warp factor satisfies the definition in Eq. (3.16)[2].

The explicit form of the metric now enables us to lower the remaining index of the three-form potential in Eq. (6.22). In principle, this is equivalent to directly using the uplift Ansatz in Eq. (4.49). One finds

$$A_{mnp} = \frac{\sqrt{2}\tan\alpha}{72 b_0}\frac{vs}{c+vs}\Big(9vs(c-vs)\xi_{[m}{}^{q}S_{np]q} + \frac{1}{12}vs(c+vs)\mathring{\eta}_{mnpqrst}\xi^{q}S^{rst} + (2c-vs)(3c-\xi vs)S_{mnp}\Big), \tag{6.30}$$

which is a slightly simplified version of the formula found in Ref. [60][3]. Note that this expression for the G_2 invariant three-form potential is totally antisymmetric, which was not manifest in the corresponding uplift Ansatz in Eq. (4.49).

The explicit form of the warp factor now enables us to compute the internal six-form potential in Eqs. (4.39, 4.40). In particular, one finds

$$\Delta A^{m} + 3\sqrt{2}\mathring{\zeta}^{m} = -\frac{3\sqrt{2}}{2m_7 b_0}\mathring{D}^{m} b_0,$$

and uses the definition for $b_0(\xi)$ in Eq. (6.28) as well as the relation for the derivative in Eq. (6.10). This finally results in

$$\Delta A^{m} = \frac{cvs}{b_0\sqrt{2}}\,\xi^{m} - 3\sqrt{2}\mathring{\zeta}^{m}, \qquad A_{m_1\cdots m_6} = \frac{\sqrt{2}cvs}{12\cdot 6!\cdot b_0}\mathring{\eta}_{m_1\cdots m_7}\xi^{m_7} - 3\sqrt{2}\mathring{\zeta}_{m_1\cdots m_6}. \tag{6.31}$$

The rest of this section is devoted to computing the internal four-form field-strength and the Freund-Rubin term of the G_2 invariant supergravity solution. First, the field-strength Ansatz in Eq. (4.56) is more convenient in the form

$$F_{mnpq} = \Big[4m_7\Delta^{6}\mathring{g}_{mr_1}\left(\Delta^{-2}g_{nr_2}\right)\left(\Delta^{-2}g_{pr_3}\right)\left(\Delta^{-2}g_{qr_4}\right)\mathring{\eta}^{r_1\cdots r_7}A_{r_5r_6r_7} - 3m_7\Delta^{3} \times\left(\Delta^{-2}g_{ns}\right)A_{pqr}K^{r}{}_{m}{}^{IJ}K^{s\,KL}\left(u_{ij}{}^{IJ}+v_{ij\,IJ}\right)\left(u^{ij}{}_{KL}+v^{ij\,KL}\right)\Big]\Big|_{[mnpq]}, \tag{6.32}$$

such that one may use Eqs. (6.27, 6.29, 6.30) directly. The term involving the Killing forms and the four-dimensional scalars simplifies with Eqs. (6.19, 6.21, 5.11) to

$$K^{mn\,IJ}K^{p\,KL}\left(u_{ij}{}^{IJ}+v_{ij\,IJ}\right)\left(u^{ij}{}_{KL}+v^{ij\,KL}\right) = \frac{8}{3}cvs\,(c+vs)\,\xi^{[m}\mathring{g}^{n]p} + s^{2}\sin^{2}\alpha \times\left[12\,vs\,\xi^{[m}{}_{q}S^{np]q} - \frac{1}{9}vs\,\mathring{\eta}^{mnpqrst}\xi_{q}S_{rst} - \left(8c+\frac{4}{3}\xi vs\right)S^{mnp}\right]. \tag{6.33}$$

[2]The determinant of the metric can be computed using the second equation in Eq. (6.8) to replace ξ_{mn} in Eq. (6.27) [60].

[3]The formula for A_{mnp} above differs from the expression in Ref. [60] by a factor of $1/6$, which is due to our conventions. Appropriately, the definitions for the field-strength also differ by that factor, such that the resulting expressions for F_{mnpq} agree.

Putting all together and using the identities in Eqs. (6.8, 6.9) finally results in

$$F_{mnpq} = \frac{\sqrt{2}v^2s^2\tan\alpha}{3b_0}\, m_7 \Bigg[\left(\frac{2c-vs}{c+vs} + \frac{c^2-v^2s^2}{b_0}\right) \xi_{[m}S_{npq]} + \frac{c-vs}{vs} \times \mathring{\eta}_{mnpqrst}S^{rst} + \frac{1}{6(3-\xi)}\left(\frac{2c-vs}{c+vs} - \frac{(c-vs)^2}{b_0}\right)\xi_{[m}\mathring{\eta}_{npq]rstu}\xi^r S^{stu}\Bigg], \tag{6.34}$$

which matches exactly the expression found in Ref. [60].

A convenient form of the field-strength Ansatz for F^{mnpq} in Eq. (4.57) is given by

$$F^{mnpq} = \Delta m_7 \left(\Delta^{-1}g^{rm}\right)\Bigg[4\mathring{\eta}_r{}^{npqstu}A_{stu} + 3\Delta^3\left(\Delta^{-1}g^{tn}\right)\left(\Delta^{-1}A_{st}{}^{p}\right)K^{q\,IJ} \times K_r{}^{s\,KL}\left(u^{ij}{}_{IJ} + v^{ij\,IJ}\right)\left(u_{ij}{}^{KL} + v_{ij\,KL}\right)\Bigg]\Bigg|_{[mnpq]}, \tag{6.35}$$

such that Eqs. (6.20, 6.22, 6.29, 6.30, 6.33) may be used directly. Taking the identities in Eqs. (6.8, 6.9) into account, one then finds

$$F^{mnpq} = \frac{\sqrt{2}v^2s^2(c+vs)^3\Delta\tan\alpha}{3}\, m_7 \Bigg[\left(\frac{2c-vs}{c+vs} + \frac{c^2-v^2s^2}{b_0}\right)\xi^{[m}S^{npq]} + \frac{(c-vs)b_0}{vs(c+vs)^2} \times \mathring{\eta}^{mnpqrst}S_{rst} - \frac{1}{6(3-\xi)}\left(\frac{2c^2-5cvs+v^2s^2}{(c+vs)^2} + \frac{(c-vs)^2}{b_0}\right)\xi^{[m}\mathring{\eta}^{npq]rstu}\xi_r S_{stu}\Bigg]. \tag{6.36}$$

Finally, the Freund-Rubin term $\mathfrak{f}_{\mathrm{FR}}(x,y)$ can be computed using the uplift Ansatz in Eq. (4.64). Inserting the explicit expressions for the inverse metric (Eq. (6.20)), the warp factor (Eq. (6.29)) and the $A_{m\,ijkl}$ and $B_{m\,ijkl}$ tensors (Eqs. (6.25, 6.26)) finally results in

$$\mathfrak{f}_{\mathrm{FR}} = \sqrt{2}m_7(c+vs)^3\left[(c^2-v^2s^2) - 2b_0\left(2c^2-3cvs+2v^2s^2-3\frac{c-vs}{c+vs}\right)\right]. \tag{6.37}$$

Here, one also used the identities in Eqs. (6.8, 6.9). Note that this formula coincides with the expression obtained in Ref. [48].

The expressions in Eqs. (6.27, 6.29, 6.30, 6.31 6.34, 6.37) enable us to derive the vector Ansätze as well. All together then represent a full bosonic solution of 11-dimensional supergravity. Furthermore, if one restricts to the maximally symmetric spacetime AdS_4, one must evaluate the above Ansätze at the stationary point of the scalar potential $V(x) = V(\alpha(x),\lambda(x))$. The latter is [63]

$$V = 2g^2(c+vs)^3\left(\left(2c^2-3cvs+2v^2s^2\right)^2 - 7\left(c^2-cvs+v^2s^2\right)\frac{c-vs}{c+vs}\right), \tag{6.38}$$

and the corresponding G_2 invariant stationary point is given by [64]

$$c^2 = \frac{3+2\sqrt{3}}{5}, \qquad s^2 = \frac{2\sqrt{3}-2}{5}, \qquad v^2 = \frac{3-\sqrt{3}}{4}. \tag{6.39}$$

Hence, inserting these values into the above Ansätze yields the Freund-Rubin solution.

6.3. Verifying the G_2 Invariant Solution

This section finally verifies the G_2 invariant supergravity solution that is described in Section 6.2. Let us first test whether the internal form potentials and field-strength as well as the Freund Rubin term satisfy Eq. (5.18). Therefore, one must compute

$$\frac{\Delta^{-3}}{6}\mathring{D}_m\left(\Delta A^m\right) = -\sqrt{2}m_7(c+vs)^3\left(2b_0\left(c^2-3cvs+v^2s^2\right)+\left(c^2-v^2s^2\right)^2\right),$$
$$\frac{\Delta^{-3}}{4!\sqrt{2}}\mathring{\eta}^{m_1\cdots m_7}A_{m_1\cdots m_3}F_{m_4\cdots m_7} = \sqrt{2}m_7(c+vs)^3\left(c^2-v^2s^2-1\right)\left(6b_0\frac{c-vs}{c+vs}+c^2-v^2s^2\right).$$

Inserting these expressions into the rhs of Eq. (5.18) yields the Freund-Rubin term that was already computed in Eq. (6.37). Therefore, the corresponding uplift Ansätze are consistent.

The second test is the verification of the equations of motion (Eqs. (5.19 – 5.21)) at the G_2 invariant stationary point of the scalar potential, which is given in Eq. (6.39). Therefore, the value of the scalar potential at the stationary point and the inverse AdS_4 radius (Eqs. (6.38, 5.30)) reduce to

$$V_\star = -\frac{216}{125}\sqrt{10}\,3^{1/4}g^2, \qquad m_4^2 = \frac{144}{125}\sqrt{10}\,3^{1/4}\,m_7^2. \tag{6.40}$$

Furthermore, let us compute the Freund-Rubin term at the stationary point. Inserting Eq. (6.39) into Eq. (6.37) results in

$$\mathfrak{f}_{\mathrm{FR}}\Big|_\star = \frac{216}{125}\sqrt{5}\,3^{1/4}\,m_7. \tag{6.41}$$

In particular, it satisfies the conjecture in Eq. (4.65) — the y-dependence drops out and $\mathfrak{f}_{\mathrm{FR}}$ becomes a constant. Note that here and in the following, a subscript $\star$ always denotes that the field is evaluated at the G_2 invariant stationary point.

Finally, one computes the following expressions at the stationary point:

$$\Delta g_{mn}\Big|_\star = \frac{5}{108}\sqrt{10}\,\frac{3^{3/4}}{15-\xi}\left((33-\xi)\mathring{g}_{mn} + 6\xi_{mn}\right),$$

$$\Delta^{-3}\Big|_\star = \frac{3}{1250}\sqrt{10}\,3^{1/4}\,(15-\xi)^2, \qquad \mathring{D}_m \log\Delta\Big|_\star = \frac{4}{3}\frac{m_7}{15-\xi}\xi_m,$$

$$\mathring{D}_m\left(\Delta^{-1}g^{mn}\mathring{D}_n\log\Delta\right)\Big|_\star = \frac{144}{25}\sqrt{10}\,3^{1/4}\left(\frac{21-26\xi+\xi^2}{(15-\xi)^2}\right)m_7^2,$$

$$\mathring{D}_p\tilde{\Gamma}^p_{mn}\Big|_\star = \frac{12m_7^2}{(15-\xi)^3}\left[\left(-783+675\xi-45\xi^2+\xi^3\right)\mathring{g}_{mn} - 4\left(9-30\xi+\xi^2\right)\xi_{mn}\right],$$

$$\tilde{\Gamma}^q_{mp}\tilde{\Gamma}^p_{qn}\Big|_\star = -\frac{6(3-\xi)m_7^2}{(15-\xi)^3}\left[\left(261-18\xi+\xi^2\right)\mathring{g}_{mn} + 8(24-\xi)\xi_{mn}\right],$$

$$F_{mpqr}F_n{}^{pqr}\Big|_\star = \frac{24}{5}\frac{m_7^2}{(15-\xi)^2}\left(\left(1935-108\xi+\xi^2\right)\mathring{g}_{mn} + (270-6\xi)\,\xi_{mn}\right),$$

$$\Delta^{-1}F_{mnpq}F^{mnpq}\Big|_\star = \frac{5184}{125}\sqrt{10}\,3^{1/4}\left(\frac{35-\xi}{15-\xi}\right)m_7^2,$$

$$\begin{aligned}\frac{1}{24}\mathring{\eta}^{mnpqrst}F_{qrst}\Big|_\star = \frac{3^{3/4}m_7}{(15-\xi)^2}\Bigg(&\left(-81+36\sqrt{3}+3\xi\right)\xi_q{}^{[m}S^{np]q} + \left(\frac{17}{12}-\frac{1}{\sqrt{3}}-\frac{1}{36}\xi\right)\\ &\times\mathring{\eta}^{mnpqrst}\xi_q S_{rst} + \left(66+24\sqrt{3}+3\xi-4\sqrt{3}\xi-\frac{1}{3}\xi^2\right)S^{mnp}\Bigg),\end{aligned}$$

$$\begin{aligned}\mathring{D}_q\left(\Delta^{-1}F^{mnpq}\right)\Big|_\star = \frac{648}{125}\frac{\sqrt{10}\,m_7^2}{(15-\xi)^2}\Bigg(&\left(-81+36\sqrt{3}+3\xi\right)\xi_q{}^{[m}S^{np]q} + \left(\frac{17}{12}-\frac{1}{\sqrt{3}}-\frac{1}{36}\xi\right)\\ &\times\mathring{\eta}^{mnpqrst}\xi_q S_{rst} + \left(66+24\sqrt{3}+3\xi-4\sqrt{3}\xi-\frac{1}{3}\xi^2\right)S^{mnp}\Bigg).\end{aligned}$$

These are obtained using the identities in Eqs. (6.8, 6.9, 6.10). It is then straightforward to verify the reduced equations of motion by inserting these expressions into Eqs. (5.19 – 5.21, 5.31, 5.32). The obtained Freund-Rubin solution is hence, verified.

7. SO(3)×SO(3) Invariant Supergravity

This chapter presents the embedding of SO(3)×SO(3) invariant supergravity into 11 dimensions. In principle, it can be obtained by repeating the steps of the previous two chapters for an SO(3)×SO(3) invariant scalar field configuration. However, there are more group invariant selfdual and anti-selfdual four-tensors satisfying various identities. As a direct consequence, there are more group invariant S^7 quantities that satisfy many non-trivial identities, which are essential in order to simplify the obtained 11-dimensional fields. Without these simplifications, it would be impossible to perform the consistency checks for the obtained solution.

The first section presents the SO(3)×SO(3) invariant selfdual and anti-selfdual four-tensors, which are required to express the vacuum expectation value as well as the $u_{ij}{}^{IJ}(x)$ and $v_{ijIJ}(x)$ tensors in an appropriate way. It also gives the various relations between them and defines the corresponding S^7 quantities. The corresponding identities for those S^7 tensors are derived and collected in Appendix C.

Section 7.2 then constructs the 11-dimensional internal fields in terms of the four-dimensional ones: the metric and its inverse, the warp factor, the form potentials, the four-form field-strength and the Freund-Rubin factor. As it turns out, the intermediate expressions become too long to write them down. Hence, only the final results are presented, since the main guideline is the same as in the previous chapters. Finally, Section 7.3 verifies the obtained Freund-Rubin solution.

7.1. SO(3)×SO(3) Invariant Tensors and Corresponding S^7 Quantities

Let us repeat the steps of Sections 5.1 and 6.1 for an SO(3)×SO(3) invariant scalar field configuration. First, the selfdual and anti-selfdual SO(3)×SO(3) invariant tensors are

$$\begin{aligned} \Phi^{(0)}_{IJKL} &= Y^{+}_{IJKL} = 4!\left(\delta^{1234}_{IJKL} + \delta^{5678}_{IJKL}\right), & \Psi^{(0)}_{IJKL} &= Y^{-}_{IJKL} = 4!\left(\delta^{1235}_{IJKL} + \delta^{4678}_{IJKL}\right), \\ \Phi^{(1)}_{IJKL} &= Z^{+}_{IJKL} = 4!\left(\delta^{1235}_{IJKL} - \delta^{4678}_{IJKL}\right), & \Psi^{(1)}_{IJKL} &= Z^{-}_{IJKL} = 4!\left(\delta^{1234}_{IJKL} - \delta^{5678}_{IJKL}\right). \end{aligned} \tag{7.1}$$

The two SO(3) subgroups act on the subspaces defined by $I = 1,2,3$ and $I = 6,7,8$ respectively. Note that this implies that there is another SO(3)×SO(3) invariant two-tensor,

$$F_{IJ} = \delta^{45}_{IJ}. \tag{7.2}$$

It corresponds to an additional U(1) rotation in the subspace that is not affected by the SO(3)'s. Later, this tensor will be required to simplify some products of S^7 quantities in a convenient way.

The vacuum expectation value $\phi_{IJKL}(x)$ decomposes into the tensors defined in Eq. (7.1). With

$$\lambda^{(0)}(x) = \frac{\lambda(x)}{2}\cos\alpha(x), \qquad \mu^{(0)}(x) = \frac{\lambda(x)}{2}\cos\alpha(x),$$
$$\lambda^{(1)}(x) = -\frac{\lambda(x)}{2}\sin\alpha(x), \qquad \mu^{(1)}(x) = \frac{\lambda(x)}{2}\sin\alpha(x),$$

one finds

$$\phi_{IJKL} = \frac{\lambda}{2}\left[\cos\alpha\left(Y^{+}_{IJKL} + iY^{-}_{IJKL}\right) - \sin\alpha\left(Z^{+}_{IJKL} - iZ^{-}_{IJKL}\right)\right]. \tag{7.3}$$

Let us now look for relations between the $Y^{\pm}$ and $Z^{\pm}$ tensors. First, from Eq. (7.1) there are the quadratic identities

$$Y^+Y^+ = Z^-Z^-, \qquad Y^-Y^- = Z^+Z^+, \tag{7.4}$$
$$Y^+Z^+ = Z^-Y^-, \qquad Y^-Z^- = Z^+Y^+, \tag{7.5}$$
$$Y^+Y^- = Z^-Z^+, \qquad Y^-Y^+ = Z^+Z^-, \tag{7.6}$$
$$Z^+Y^- = Y^-Z^+, \qquad Y^+Z^- = Z^-Y^+. \tag{7.7}$$

These do not help in simplifying the sums for ${u_{ij}}^{IJ}(x)$ and $v_{ijIJ}(x)$ in Eq. (5.4) but in Appendix C, they will be useful to find convenient identities between the corresponding S^7 quantities. Furthermore, there are cubic identities between the SO(8) tensors defined in Eq. (7.1). Indeed,

$$\left(Y^+\right)^3 = 4Y^+, \qquad \left(Z^+\right)^3 = 4Z^+, \tag{7.8}$$

$$\begin{gathered}\left(Y^-\right)^3 = 4Y^-, \qquad Y^+Y^-Y^+ = 0, \qquad Y^-Y^+Y^- = 0,\\ Y^-Y^+Y^+ + Y^+Y^+Y^- = 4Y^-, \qquad Y^+Y^-Y^- + Y^-Y^-Y^+ = 4Y^+,\end{gathered} \tag{7.9}$$

$$\begin{gathered}\left(Z^-\right)^3 = 4Z^-, \qquad Z^+Z^-Z^+ = 0, \qquad Z^-Z^+Z^- = 0,\\ Z^-Z^+Z^+ + Z^+Z^+Z^- = 4Z^-, \qquad Z^+Z^-Z^- + Z^-Z^-Z^+ = 4Z^+.\end{gathered} \tag{7.10}$$

With these cubic relations, one may now construct a list of all group invariant four-tensors that are required in order to simplify the ${u_{ij}}^{IJ}(x)$ and $v_{ijIJ}(x)$ tensors in Eq. (5.4).

It turns out that this list only contains the selfdual and anti-selfdual tensors $Y^{\pm}$ and $Z^{\pm}$ as well as

$$\Pi = \frac{1}{8}\left(Y^+ + iY^-\right)\left(Y^+ - iY^-\right) = \frac{1}{8}\left(Z^+ - iZ^-\right)\left(Z^+ + iZ^-\right). \tag{7.11}$$

In particular, the latter satisfies the convenient properties

$$\Pi^2 = \Pi, \qquad \Pi^*_{IJKL} = \Pi_{KLIJ}, \tag{7.12}$$
$$\left(Y^+ - iY^-\right)\Pi = Y^+ - iY^-, \qquad \left(Z^+ + iZ^-\right)\Pi = Z^+ + iZ^-, \tag{7.13}$$
$$\phi\phi^* = 2\lambda^2\Pi, \qquad \phi^*\Pi = \phi^*, \tag{7.14}$$

which can be proved using the cubic identities in Eqs. (7.9, 7.10). Finally, Eq. (5.4) reduces to

$$u_{IJ}{}^{KL} = \delta^{KL}_{IJ} + (\tilde{c} - 1)\Pi_{IJKL}, \tag{7.15}$$
$$v^{IJKL} = \frac{\tilde{s}}{2\sqrt{2}}\left[\cos\alpha(Y^+ - iY^-) - \sin\alpha(Z^+ + iZ^-)\right]_{IJKL}, \tag{7.16}$$

where

$$\tilde{c} = \cosh(\sqrt{2}\lambda), \qquad \tilde{s} = \sinh(\sqrt{2}\lambda).$$

The explicit SO(3)×SO(3) invariant S^7 quantities are denoted as

$$\begin{aligned} \xi^{(0)} &= \xi, & \xi^{(0)}_m &= \xi_m, & \xi^{(0)}_{mn} &= \xi_{mn}, & S^{(0)}_{mnp} &= S_{mnp}, \\ \xi^{(1)} &= \zeta, & \xi^{(1)}_m &= \zeta_m, & \xi^{(1)}_{mn} &= \zeta_{mn}, & S^{(1)}_{mnp} &= T_{mnp}, \end{aligned}$$

and Eqs. (5.8 – 5.10) read

$$\begin{aligned} Y^+_{IJKL} &= \frac{\xi}{6}K_m{}^{[IJ}K^{m\,KL]} - \frac{3}{2}\xi^{mn}K_m{}^{[IJ}K_n{}^{KL]} + \frac{1}{12}\xi^m K_{mn}{}^{[IJ}K^{n\,KL]}, \\ Y^-_{IJKL} &= \frac{1}{2}S^{mnp}K_{mn}{}^{[IJ}K_p{}^{KL]}, \end{aligned} \tag{7.17}$$

$$\begin{aligned} Z^+_{IJKL} &= \frac{\zeta}{6}K_m{}^{[IJ}K^{m\,KL]} - \frac{3}{2}\zeta^{mn}K_m{}^{[IJ}K_n{}^{KL]} + \frac{1}{12}\zeta^m K_{mn}{}^{[IJ}K^{n\,KL]}, \\ Z^-_{IJKL} &= \frac{1}{2}T^{mnp}K_{mn}{}^{[IJ}K_p{}^{KL]}, \end{aligned} \tag{7.18}$$

$$\begin{aligned} \xi_m = \frac{1}{16}Y^+_{IJKL}K^{IJ}_{mn}K^{n\,KL}\,, \quad \xi_{mn} &= -\frac{1}{16}Y^+_{IJKL}K^{IJ}_m K^{KL}_n\,, \quad \xi = \mathring{g}^{mn}\xi_{mn}, \\ S_{mnp} &= \frac{1}{16}Y^-_{IJKL}K_{[mn}{}^{IJ}K_{p]}{}^{KL}. \end{aligned} \tag{7.19}$$

$$\begin{aligned} \zeta_m = \frac{1}{16}Z^+_{IJKL}K^{IJ}_{mn}K^{n\,KL}\,, \quad \zeta_{mn} &= -\frac{1}{16}Z^+_{IJKL}K^{IJ}_m K^{KL}_n\,, \quad \zeta = \mathring{g}^{mn}\zeta_{mn}, \\ T_{mnp} &= \frac{1}{16}Z^-_{IJKL}K_{[mn}{}^{IJ}K_{p]}{}^{KL}. \end{aligned} \tag{7.20}$$

In addition, one may also decompose the F_{IJ} tensor into the basis of antisymmetric matrices provided by the Killing forms,

$$F_{IJ} = \frac{1}{8} F_m K^{m\,IJ} + \frac{1}{16} F_{mn} K^{mn\,IJ}, \qquad F_m = F_{IJ} K_m{}^{IJ}, \qquad F_{mn} = F_{IJ} K_{mn}{}^{IJ}. \tag{7.21}$$

The derivation of the various identities for these tensors is given in Appendix C. Furthermore, the generic relations in Eqs. (5.12 – 5.16) translate to

$$\begin{aligned} \mathring{D}_m \xi &= 2m_7 \xi_m, & \mathring{D}_m \xi_n &= 6m_7\, \xi_{mn} - 2m_7\, \xi \mathring{g}_{mn}, \\ \mathring{D}_m \xi_{np} &= \frac{1}{3} m_7 \Big(\mathring{g}_{np} \xi_m - \mathring{g}_{m(n} \xi_{p)} \Big), & \mathring{D}_m S_{npq} &= \frac{1}{6} m_7 \mathring{\eta}_{mnpq}{}^{rst} S_{rst}, \end{aligned} \tag{7.22}$$

$$\begin{aligned} \mathring{D}_m \zeta &= 2m_7 \zeta_m, & \mathring{D}_m \zeta_n &= 6m_7\, \zeta_{mn} - 2m_7\, \zeta \mathring{g}_{mn}, \\ \mathring{D}_m \zeta_{np} &= \frac{1}{3} m_7 \Big(\mathring{g}_{np} \zeta_m - \mathring{g}_{m(n} \zeta_{p)} \Big), & \mathring{D}_m T_{npq} &= \frac{1}{6} m_7 \mathring{\eta}_{mnpq}{}^{rst} T_{rst}, \end{aligned} \tag{7.23}$$

$$\mathring{D}_m F_n = -m_7 F_{mn}, \qquad \mathring{D}_p F_{mn} = 2m_7 \mathring{g}_{p[m} F_{n]} \tag{7.24}$$

and

$$Y_+^{IJKL} K_m{}^{KL} = -2\xi_{mn} K^{n\,IJ} - \frac{1}{3} \xi^n K_{mn}{}^{IJ}, \tag{7.25}$$

$$Y_+^{IJKL} K_{mn}{}^{KL} = \frac{2}{3} \xi_{[m} K_{n]}{}^{IJ} + \left(\frac{2}{3} \xi \mathring{g}_{mp} \mathring{g}_{nq} - 4 \mathring{g}_{p[m} \xi_{n]q} \right) K^{pq\,IJ}, \tag{7.26}$$

$$Y_-^{IJKL} K_m{}^{KL} = S_{mnp} K^{np\,IJ}, \tag{7.27}$$

$$Y_-^{IJKL} K_{mn}{}^{KL} = 2S_{mnp} K^{p\,IJ} - \frac{1}{6} \mathring{\eta}_{mn}{}^{p_1 \cdots p_5} S_{p_1 p_2 p_3} K_{p_4 p_5}{}^{IJ}, \tag{7.28}$$

$$Z_+^{IJKL} K_m{}^{KL} = -2\zeta_{mn} K^{n\,IJ} - \frac{1}{3} \zeta^n K_{mn}{}^{IJ}, \tag{7.29}$$

$$Z_+^{IJKL} K_{mn}{}^{KL} = \frac{2}{3} \zeta_{[m} K_{n]}{}^{IJ} + \left(\frac{2}{3} \zeta \mathring{g}_{mp} \mathring{g}_{nq} - 4 \mathring{g}_{p[m} \zeta_{n]q} \right) K^{pq\,IJ}, \tag{7.30}$$

$$Z_-^{IJKL} K_m{}^{KL} = T_{mnp} K^{np\,IJ}, \tag{7.31}$$

$$Z_-^{IJKL} K_{mn}{}^{KL} = 2T_{mnp} K^{p\,IJ} - \frac{1}{6} \mathring{\eta}_{mn}{}^{p_1 \cdots p_5} T_{p_1 p_2 p_3} K_{p_4 p_5}{}^{IJ}. \tag{7.32}$$

Note that Eq. (7.24) can directly be obtained from the definition in Eq. (7.21) and Eq. (4.11).

Finally, the corresponding contraction identities for the Π tensor read

$$\begin{aligned} \Pi_{IJKL} K_m{}^{KL} = \frac{1}{36} \Big[\Big(&(18 - \xi^2 - \zeta^2) \mathring{g}_{mn} + 6\xi \xi_{mn} + 6\zeta \zeta_{mn} - \xi_m \xi_n - \zeta_m \zeta_n \\ &+ 3i S_{mnp} \xi^p - 3i T_{mnp} \zeta^p \Big) K^{n\,IJ} + \Big(6\xi_{mn} \xi_p + 6\zeta_{mn} \zeta_p - \xi \mathring{g}_{mn} \xi_p \\ &- \zeta \mathring{g}_{mn} \zeta_p - 3i\xi S_{mnp} + 3i\zeta T_{mnp} + 18i S_{mn}{}^q \xi_{pq} - 18i T_{mn}{}^q \zeta_{pq} \Big) K^{np\,IJ} \Big], \end{aligned} \tag{7.33}$$

$$\Pi_{IJKL}K_{mn}{}^{KL} = \frac{1}{18}\Big[\xi\xi_{[m}\mathring{g}_{n]p} + \zeta\zeta_{[m}\mathring{g}_{n]p} - 6\xi_{[m}\xi_{n]p} - 6\zeta_{[m}\zeta_{n]p} + 3i\xi S_{mnp} - 3i\zeta T_{mnp}$$
$$- 18i\xi^{q}{}_{[m}S_{n]pq} + 18i\zeta^{q}{}_{[m}T_{n]pq}\Big]K^{p\,IJ} + \frac{1}{36}\Big[(18+\xi^2+\zeta^2)\delta^{pq}_{mn}$$
$$- 12\xi\xi^{p}{}_{[m}\delta^{q}{}_{n]} - 12\zeta\zeta^{p}{}_{[m}\delta^{q}{}_{n]} + 36\xi^{p}{}_{m}\xi^{q}{}_{n} + 36\zeta^{p}{}_{m}\zeta^{q}{}_{n}\Big]K_{pq}{}^{IJ}$$
$$+ \frac{i}{24}\Big[\xi_{[m}S_{n]pq} - \xi_{[p}S_{q]mn} - \zeta_{[m}T_{n]pq} + \zeta_{[p}T_{q]mn} - \xi^{r}{}_{[m}\mathring{\eta}_{n]pqrstu}S^{stu}$$
$$+ \xi^{r}{}_{[p}\mathring{\eta}_{q]mnrstu}S^{stu} + \zeta^{r}{}_{[m}\mathring{\eta}_{n]pqrstu}T^{stu} - \zeta^{r}{}_{[p}\mathring{\eta}_{q]mnrstu}T^{stu}\Big]K^{pq\,IJ}. \quad (7.34)$$

7.2. Constructing the SO(3)×SO(3) Invariant Supergravity Solution

This section constructs the SO(3)×SO(3) invariant 11-dimensional supergravity. This requires the contraction relations of the previous section as well as the S^7 tensor identities derived in Appendix C. Since the intermediate expressions (e.g. in Eqs. (7.33, 7.34)) become rather long, it is convenient to use the following trick in deriving the final 11-dimensional fields: The uplift formulae in Eqs. (4.30, 4.31, 4.45, 4.64) always contain double contractions between the $u_{ij}{}^{IJ}(x)$ and $v_{ijIJ}(x)$ tensors. Therefore, one may reduce the order of S^7 tensors in all Ansätze via

$$u_{MN}{}^{IJ}u^{MN}{}_{KL} = \delta^{IJ}_{KL} + \tilde{s}^2\Pi^*_{IJKL}, \qquad v_{MNIJ}v^{MNKL} = \tilde{s}^2\Pi_{IJKL}, \quad (7.35)$$

$$u_{MN}{}^{IJ}v^{MNKL} = \frac{\tilde{s}\tilde{c}}{2\sqrt{2}}\left[\cos\alpha(Y^+ - iY^-) - \sin\alpha(Z^+ + iZ^-)\right]_{IJKL}, \quad (7.36)$$

$$v_{MNIJ}u^{MN}{}_{KL} = \frac{\tilde{s}\tilde{c}}{2\sqrt{2}}\left[\cos\alpha(Y^+ + iY^-) - \sin\alpha(Z^+ - iZ^-)\right]_{IJKL}. \quad (7.37)$$

These relations can be proved using Eqs. (7.11 – 7.16). Then, one may simply find the inverse internal metric by inserting Eqs. (7.35 – 7.37) into the Ansatz in Eq. (4.30), and using the contraction identities in Eqs. (7.25 – 7.34) and Eq. (5.11). One finally finds

$$\Delta^{-1}g^{mn} = \left[\tilde{c}^2 - \frac{\tilde{s}^2}{18}(\xi^2+\zeta^2)\right]\mathring{g}^{mn} - \frac{\tilde{s}^2}{18}(\xi^m\xi^n + \zeta^m\zeta^n) + \frac{\tilde{s}}{3}\left(\mathcal{X}_1\xi^{mn} + \mathcal{Z}_1\zeta^{mn}\right), \quad (7.38)$$

where

$$\mathcal{X}_1(\alpha) = \xi\tilde{s} - 3\sqrt{2}\tilde{c}\cos\alpha, \qquad \mathcal{Z}_1(\alpha) = \zeta\tilde{s} + 3\sqrt{2}\tilde{c}\sin\alpha. \quad (7.39)$$

The result has been simplified using the S^7 tensor identities in Appendix C.

In the same way, Eq. (4.31) reduces to

$$\Delta^{-1}A_{mn}{}^{p} = \frac{\tilde{s}}{18\sqrt{2}}\left(6\tilde{s}\xi_{q[m}S_{n]}{}^{pq} - 6\tilde{s}\zeta_{q[m}T_{n]}{}^{pq} - \mathcal{X}_1 S_{mn}{}^{p} + \mathcal{Z}_1 T_{mn}{}^{p}\right). \quad (7.40)$$

For a convenient metric expression, one must use Eqs. (7.35 – 7.37) and Eq. (5.11) twice as the corresponding Ansatz in Eq. (4.45) is quartic in the ${u_{ij}}^{IJ}(x)$ and $v_{ijIJ}(x)$ tensors. Using the relations in Appendix C finally yields

$$\Delta^{-2} g_{mn} = \frac{1}{18}(\mathcal{X}_1^2 + \mathcal{Z}_1^2)\mathring{g}_{mn} - \frac{1}{3}\tilde{s}(\mathcal{X}_1 \xi_{mn} + \mathcal{Z}_1 \zeta_{mn}) + \frac{\tilde{s}^2}{36} f_m f_n, \tag{7.41}$$

where

$$f_m(\alpha) = \sqrt{2}\tilde{c}\,(\sin\alpha\,\xi_m + \cos\alpha\,\zeta_m) + 3\tilde{s}F_m. \tag{7.42}$$

One may now read off the warp factor. Indeed, $\Delta^{-2} g_{mn}(x,y)$ is inverse to the expression in Eq. (7.38),

$$\Delta^{-1} g^{mp} \Delta^{-2} g_{pn} = \Delta^{-3} \delta_n^m, \tag{7.43}$$

for

$$\Delta^{-3} = \frac{1}{36}\left(\mathcal{X}_2^2 + 2\tilde{c}^2 \mathcal{X}_2 \mathcal{Z}_2 + \mathcal{Z}_2^2 + \mathcal{Y}\right). \tag{7.44}$$

Here, one abbreviates

$$\begin{gathered} \mathcal{X}_2(\alpha) = \sqrt{2}\cos\alpha\,\xi\tilde{s} - 3\tilde{c}\,, \qquad \mathcal{Z}_2(\alpha) = -\sqrt{2}\sin\alpha\,\zeta\tilde{s} - 3\tilde{c}\,, \\ \mathcal{Y}(\alpha) = \tilde{s}^4(\cos^2\alpha - \sin^2\alpha)(\xi^2 - \zeta^2). \end{gathered} \tag{7.45}$$

Note that Eq. (7.44) exactly coincides with the expression found in Ref. [51], where the warp factor has been computed by taking the determinant of the metric expression in Eq. (7.41).

The remaining upper index of the three-form potential in Eq. (7.40) may finally be lowered with the metric,

$$\begin{aligned} \Delta^{-3} A_{mnp} = \frac{1}{18\sqrt{2}} \Bigg[&-\left(\frac{\tilde{s}}{2}(1+\tilde{c}^2)\mathcal{X}_1 + \frac{\tilde{s}^3\tilde{c}}{6}\sqrt{2}\,\zeta\,(\sin\alpha\,\xi + \cos\alpha\,\zeta)\right) S_{mnp} \\ &+ \left(\frac{\tilde{s}}{2}(1+\tilde{c}^2)\mathcal{Z}_1 - \frac{\tilde{s}^3\tilde{c}}{6}\sqrt{2}\,\xi\,(\sin\alpha\,\xi + \cos\alpha\,\zeta)\right) T_{mnp} \\ &+ \frac{\tilde{s}^2}{108}\mathring{\eta}_{mnpqrst}\,(\mathcal{Z}_1\xi^q - \mathcal{X}_1\zeta^q)\left(\mathcal{Z}_1 S^{rst} + \mathcal{X}_1 T^{rst}\right) \\ &- \frac{\tilde{s}^3}{12}\left(\mathcal{X}_1 \xi_{[m} + \mathcal{Z}_1 \zeta_{[m}\right)\left(S_{np]q}\xi^q - T_{np]q}\zeta^q\right)\Bigg], \end{aligned} \tag{7.46}$$

and the dual six-form potential (Eq. (4.39)) can be computed using Eq. (7.22, 7.23),

$$\begin{aligned} \Delta A^m = -\frac{\tilde{s}\Delta^3}{6\sqrt{2}} \Bigg[&\left((1+\tilde{c}^2\cos 2\alpha)\mathcal{X}_1 - \tilde{c}^2 \sin 2\alpha\,\mathcal{Z}_1\right)\xi^m \\ &+ \left(-\tilde{c}^2\sin 2\alpha\,\mathcal{X}_1 + (1-\tilde{c}^2\cos 2\alpha)\mathcal{Z}_1\right)\zeta^m\Bigg] - 3\sqrt{2}\mathring{\zeta}^m. \end{aligned} \tag{7.47}$$

Both expressions have been simplified using the S^7 tensor identities in Appendix C. Finally, the six-form is obtained by dualizing the expression in Eq. (7.47),

$$A_{m_1\cdots m_6} = -\frac{\tilde{s}\Delta^3}{36\cdot 6!\sqrt{2}}\mathring{\eta}_{m_1\cdots m_7}\Big[\Big((1+\tilde{c}^2\cos 2\alpha)\mathcal{X}_1 - \tilde{c}^2\sin 2\alpha\,\mathcal{Z}_1\Big)\xi^{m_7} + \Big(-\tilde{c}^2\sin 2\alpha\,\mathcal{X}_1 + (1-\tilde{c}^2\cos 2\alpha)\mathcal{Z}_1\Big)\zeta^{m_7}\Big] - 3\sqrt{2}\mathring{\zeta}_{m_1\cdots m_6}. \tag{7.48}$$

In order to find the four-form field-strength, one must first compute the expression in Eq. (6.33) for the SO(3)×SO(3) invariant vacuum expectation value. Again, using Eqs. (7.35 – 7.37) and Eq. (5.11) as well as the identities in Appendix C, one finally finds

$$K^{mn\,IJ}K^{p\,KL}\left(u_{ij}{}^{IJ} + v_{ij\,IJ}\right)\left(u^{ij}{}_{KL} + v^{ij\,KL}\right) = \frac{8}{9}\tilde{s}\Big[\left(\xi\tilde{s} + 3\sqrt{2}\tilde{c}\cos\alpha\right)\xi^{[m}\mathring{g}^{n]p} + \left(\zeta\tilde{s} - 3\sqrt{2}\tilde{c}\sin\alpha\right)\zeta^{[m}\mathring{g}^{n]p} - 6\tilde{s}\left(\xi^{[m}\xi^{n]p} + \zeta^{[m}\zeta^{n]p}\right)\Big]. \tag{7.49}$$

This can be used to derive the four-form via Eq. (6.32). The final simplified result is

$$F_{mnpq} = \frac{m_7\Delta^6}{36^2}\Big[\mathring{\eta}_{mnpqrst}\left(c_S S^{rst} + c_T T^{rst} + c_U f^r S^{stu}\xi_u\right) + c_V\left(\mathcal{Z}_1\xi_{[m} - \mathcal{X}_1\zeta_{[m}\right)\left(\mathcal{Z}_1 S_{npq]} + \mathcal{X}_1 T_{npq}\right)\Big] + \frac{m_7\Delta^3}{36}\left(c_{XS}\,\xi_{[m}S_{npq]} + c_{XT}\,\xi_{[m}T_{npq]} + c_{ZS}\,\zeta_{[m}S_{npq]} + c_{ZT}\,\zeta_{[m}T_{npq]}\right), \tag{7.50}$$

where one abbreviates

$$\begin{aligned}
c_S &= \frac{2sc\cos\alpha}{3}\left(\mathcal{X}_1^2 + \mathcal{Z}_1^2\right)\left((1+c^2\cos 2\alpha)\mathcal{X}_1^2 + (1-c^2\cos 2\alpha)\mathcal{Z}_1^2 - 2c^2\sin 2\alpha\mathcal{X}_1\mathcal{Z}_1\right)\\
&\quad + 2\sqrt{2}s\left(-(1+c^2\cos 2\alpha)\mathcal{X}_1^3 - (1+3c^2\cos 2\alpha)\mathcal{X}_1\mathcal{Z}_1^2 + 2c^2\sin 2\alpha\mathcal{Z}_1^3\right),\\
c_T &= \frac{2sc\sin\alpha}{3}\left(\mathcal{X}_1^2 + \mathcal{Z}_1^2\right)\left((1+c^2\cos 2\alpha)\mathcal{X}_1^2 + (1-c^2\cos 2\alpha)\mathcal{Z}_1^2 - 2c^2\sin 2\alpha\mathcal{X}_1\mathcal{Z}_1\right)\\
&\quad + 2\sqrt{2}s\left((1-c^2\cos 2\alpha)\mathcal{Z}_1^3 + (1-3c^2\cos 2\alpha)\mathcal{X}_1^2\mathcal{Z}_1 - 2c^2\sin 2\alpha\mathcal{X}_1^3\right),\\
c_U &= 2\sqrt{2}s^3c^2\left(\mathcal{Z}_1^2\sin 2\alpha - 2\mathcal{X}_1\mathcal{Z}_1\cos 2\alpha - \mathcal{X}_1^2\sin 2\alpha\right),\\
c_V &= 16\sqrt{2}s^2(c^2+1)\left(c\mathcal{X}_2 + c\mathcal{Z}_2 + 3s^2\right),\\
c_{XS} &= -16s^2c\cos\alpha\mathcal{X}_1 + 16s^2c\sin\alpha\mathcal{Z}_1 + 24\sqrt{2}s^2(1+c^2\cos 2\alpha),\\
c_{XT} &= -16s^2c\sin\alpha\mathcal{X}_1 - 16s^2c\cos\alpha\mathcal{Z}_1 + 24\sqrt{2}s^2c^2\sin 2\alpha),\\
c_{ZS} &= -16s^2c\sin\alpha\mathcal{X}_1 - 16s^2c\cos\alpha\mathcal{Z}_1 - 24\sqrt{2}s^2c^2\sin 2\alpha),\\
c_{ZT} &= +16s^2c\cos\alpha\mathcal{X}_1 - 16s^2c\sin\alpha\mathcal{Z}_1 - 24\sqrt{2}s^2(1-c^2\cos 2\alpha).
\end{aligned}$$

The expression for $\Delta^{-1}F^{mnpq}(x,y)$ has the same structure but contains even more terms. Therefore, it is not displayed in full generality here, since it is only required in order to verify the Freund-Rubin solution. Hence, it is computed in the next section at the stationary point of the scalar potential, where it simplifies further.

Finally, one may compute the Freund-Rubin term using the Ansatz in Eq. (4.64). One performs the same steps as for the derivation of the internal metric. The final result is

$$\mathfrak{f}_{\mathrm{FR}} = m_7 \left[\frac{\sqrt{2}}{4} \left(12 + 8\tilde{s}^2 - \tilde{s}^4\right) - \frac{\tilde{s}\tilde{c}}{6} \left(4 - \tilde{s}^2\right) \left(\xi \cos\alpha - \zeta \sin\alpha\right) \right]. \tag{7.51}$$

In particular, it satisfies the conjecture in Eq. (4.65) [48].

The next section verifies the obtained solution in the case of a maximally symmetric spacetime. Therefore, all fields must be evaluated at the stationary point of the scalar potential [51]

$$V = \frac{g^2}{2}(\tilde{s}^4 - 8\tilde{s}^2 - 12), \tag{7.52}$$

which is given by

$$\tilde{c} = \sqrt{5}, \qquad \tilde{s} = 2. \tag{7.53}$$

Note that the stationary point does not depend on α. The reason is that a choice of α corresponds to a particular gauge of the U(1) symmetry that is generated by the two-tensor F_{IJ} [51]. Hence, one may choose α arbitrarily. Here, a symmetric value of

$$\alpha = -\frac{\pi}{4}, \qquad \sin\alpha = -\frac{1}{\sqrt{2}}, \qquad \cos\alpha = \frac{1}{\sqrt{2}} \tag{7.54}$$

is chosen, such that the α-dependent scalars $\mathcal{X}_{1,2}(\alpha)$, $\mathcal{Z}_{1,2}(\alpha)$ and $\mathcal{Y}(\alpha)$ reduce to

$$\mathcal{Y} = 0, \qquad \mathcal{X}_1 = \mathcal{X}_2 \equiv \mathcal{X} = 2\xi - 3\sqrt{5}, \qquad \mathcal{Z}_1 = \mathcal{Z}_2 \equiv \mathcal{Z} = 2\zeta - 3\sqrt{5} \tag{7.55}$$

at the stationary point. These simplifications will be used in the next section in order to verify the Freund-Rubin solution.

7.3. Verifying the SO(3)×SO(3) Invariant Solution

Let us finally verify the obtained SO(3)×SO(3) invariant solution of 11-dimensional supergravity. Therefore, one may first check, whether the fields in Eqs. (7.44, 7.46, 7.47, 7.50 7.51) satisfy the non-trivial relation in Eq. (5.18). Computing

$$\frac{\Delta^{-3}}{6}\mathring{D}_m\left(\Delta A^m\right) = m_7\sqrt{2}(1+c^2)\left(\frac{c}{6}(\mathcal{X}_2+\mathcal{Z}_2) - \frac{1}{1+c^2} - \frac{\Delta^3}{36}s^2\left(\mathcal{X}_1^2+\mathcal{Z}_1^2\right)\right),$$
$$\frac{\Delta^{-3}}{4!\sqrt{2}}\mathring{\eta}^{m_1\cdots m_7}A_{m_1\cdots m_3}F_{m_4\cdots m_7} = -m_7\sqrt{2}(1+c^2)$$
$$\times\left(\frac{s^2c}{4+4c^2}(\mathcal{X}_2+\mathcal{Z}_2) + \frac{s^2}{4} - \frac{\Delta^3}{36}s^2\left(\mathcal{X}_1^2+\mathcal{Z}_1^2\right)\right),$$

and inserting these expressions into the rhs of Eq. (5.18) indeed yields the Freund-Rubin term that was computed in Eq. (7.51). Hence, the obtained 11-dimensional fields pass this non-trivial consistency test.

The rest of this section verifies the obtained solution for a maximally symmetric space-time. Therefore, one must evaluate all fields at the stationary point of the scalar potential. In the following, this is always denoted by a subscript $\star$[1]. The scalar potential in Eq. (7.52) reduces to

$$V_\star = -14g^2, \tag{7.56}$$

and the inverse AdS$_4$ radius is given by Eq. (5.30),

$$m_4^2 = \frac{28}{3}m_7^2. \tag{7.57}$$

Let us now verify that the obtained fields satisfy the Maxwell equations in Eq. (5.21). Therefore, the Freund-Rubin parameter in Eq. (7.51) simplifies to

$$\mathfrak{f}_{\mathrm{FR}}\Big|_\star = 7\sqrt{2}m_7. \tag{7.58}$$

It satisfies the conjecture in Eq. (4.65). Furthermore, the seven-dimensional dual of the four-form field-strength in Eq. (7.50) can be computed using the S^7 tensor identities in Appendix C. At the stationary point, the resulting expression reads

$$\begin{aligned}\frac{1}{24}\mathring{\eta}^{mnpqrst}F_{qrst}\Big|_\star = \frac{\sqrt{2}m_7\Delta^6|_\star}{36^2}\Big[&d_1(\mathcal{X},\mathcal{Z})\left(\sqrt{5}\zeta^{[m}-\sqrt{5}\xi^{[m}+6F^{[m}\right)S^{np]q}\xi_q\\ &+\mathring{\eta}^{mnpqrst}\Big(d_2(\mathcal{X},\mathcal{Z})\xi_qS_{rst}+d_3(\mathcal{X},\mathcal{Z})\xi_qT_{rst}-d_3(\mathcal{Z},\mathcal{X})\zeta_qS_{rst}\\ &-d_2(\mathcal{Z},\mathcal{X})\zeta_qT_{rst}\Big)+d_4(\mathcal{X},\mathcal{Z})S^{mnp}-d_4(\mathcal{Z},\mathcal{X})T^{mnp}\Big],\end{aligned} \tag{7.59}$$

[1]The $\star$ also sets the parameter α to the value in Eq. (7.54)

where

$$
\begin{aligned}
d_1(\mathcal{X},\mathcal{Z}) &= 480\left(\mathcal{X}^2-\mathcal{Z}^2\right),\\
d_2(\mathcal{X},\mathcal{Z}) &= 4\left(\mathcal{X}^2+10\mathcal{X}\mathcal{Z}+49\mathcal{Z}^2\right)-\frac{4\sqrt{5}}{3}\left(\mathcal{X}^2-\mathcal{Z}^2\right)(\mathcal{X}+11\mathcal{Z}),\\
d_3(\mathcal{X},\mathcal{Z}) &= -4\left(5\mathcal{X}^2+2\mathcal{X}\mathcal{Z}+5\mathcal{Z}^2\right)+\frac{4\sqrt{5}}{3}\left(\mathcal{X}^3+21\mathcal{X}^2\mathcal{Z}+3\mathcal{X}\mathcal{Z}^2-\mathcal{Z}^3\right),\\
d_4(\mathcal{X},\mathcal{Z}) &= -24(\mathcal{X}+2\mathcal{Z})\left(\mathcal{X}^2-2\mathcal{X}\mathcal{Z}+5\mathcal{Z}^2\right)+4\sqrt{5}\left(\mathcal{X}^2+\mathcal{Z}^2\right)\left(\mathcal{X}^2+10\mathcal{X}\mathcal{Z}+\mathcal{Z}^2\right),
\end{aligned}
$$

and the warp factor is given by

$$
\Delta^{-3}\Big|_\star = 15-2\sqrt{5}(\xi+\zeta)+\frac{\xi^2+10\xi\zeta+\zeta^2}{9}. \tag{7.60}
$$

For the lhs of the Maxwell equations in Eq. (5.21), one must compute the four-form with upper indices using Eq. (6.35) and the S^7 tensor identities in Appendix C. In particular, at the stationary point, one finds

$$
\begin{aligned}
\Delta^{-1}F^{mnpq}\Big|_\star = \frac{\sqrt{2}m_7\Delta^3|_\star}{36}\Big[&\mathring{\eta}^{mnpqrst}\Big(d_5(\mathcal{X},\mathcal{Z})S_{rst}-d_5(\mathcal{Z},\mathcal{X})T_{rst}\\
&+d_6(\mathcal{X},\mathcal{Z})\xi_r S_{stu}\xi^u-d_6(\mathcal{Z},\mathcal{X})\zeta_r T_{stu}\zeta^u\Big)+d_7(\mathcal{X},\mathcal{Z})\xi^{[m}S^{npq]}\\
&-d_7(\mathcal{Z},\mathcal{X})\zeta^{[m}T^{npq]}+d_8(\mathcal{X},\mathcal{Z})\xi^{[m}T^{npq]}-d_8(\mathcal{Z},\mathcal{X})\zeta^{[m}S^{npq]}\Big],
\end{aligned} \tag{7.61}
$$

with

$$
\begin{aligned}
d_5(\mathcal{X},\mathcal{Z}) &= -\frac{2}{9}\left(\mathcal{X}^3+10\mathcal{X}^2\mathcal{Z}+\mathcal{X}\mathcal{Z}^2+\frac{3\sqrt{5}}{2}\left(3\mathcal{X}^2+16\mathcal{X}\mathcal{Z}+17\mathcal{Z}^2\right)+63(\mathcal{X}+5\mathcal{Z})\right),\\
d_6(\mathcal{X},\mathcal{Z}) &= +\frac{2}{9}\left(-84\mathcal{X}+\sqrt{5}\left(\mathcal{X}^2+10\mathcal{X}\mathcal{Z}+\mathcal{Z}^2\right)\right),\\
d_7(\mathcal{X},\mathcal{Z}) &= -\frac{8}{9}\left(3\left(\mathcal{X}^2+10\mathcal{X}\mathcal{Z}-27\mathcal{Z}^2\right)+\sqrt{5}\left(\mathcal{X}^2\mathcal{Z}+10\mathcal{X}\mathcal{Z}^2+\mathcal{Z}^3\right)\right),\\
d_8(\mathcal{X},\mathcal{Z}) &= -\frac{8}{9}\left(3\left(5\mathcal{X}^2+22\mathcal{X}\mathcal{Z}+5\mathcal{Z}^2\right)+\sqrt{5}\left(\mathcal{X}^2\mathcal{Z}+10\mathcal{X}\mathcal{Z}^2+\mathcal{Z}^3\right)\right).
\end{aligned}
$$

Finally, one may compute the derivative of the expression in Eq. (7.61) using the generic identities in Eqs. (7.22 – 7.24). Simplifying the result using the S^7 tensor identities in Appendix C finally yields the rhs of the Maxwell equations in Eq. (5.21), which is given by Eqs. (7.58, 7.59).

Furthermore, in order to check whether the obtained 11-dimensional fields satisfy Eq. (5.19), one must compute

$$\mathring{D}_m \log \Delta\Big|_\star = \frac{4m_7\Delta^3|_\star}{27}\left[\left(9\sqrt{5}-\xi-5\zeta\right)\xi_m + \left(9\sqrt{5}-5\xi-\zeta\right)\zeta_m\right], \tag{7.62}$$

$$\begin{aligned}\mathring{D}_m\left(\Delta^{-1}g^{mn}\mathring{D}_n\log\Delta\right)\Big|_\star = -\frac{m_7^2\Delta^6|_\star}{36^2}\Big[&3360\left(5\mathcal{X}^2+2\mathcal{X}\mathcal{Z}+5\mathcal{Z}^2\right)\\ &+64\sqrt{5}(\mathcal{X}+\mathcal{Z})\left(19\mathcal{X}^2-50\mathcal{X}\mathcal{Z}+19\mathcal{Z}^2\right)\\ &+\frac{8}{3}\left(91\mathcal{X}^4-140\mathcal{X}^3\mathcal{Z}-718\mathcal{X}^2\mathcal{Z}^2-140\mathcal{X}\mathcal{Z}^3+91\mathcal{Z}^4\right)\Big],\end{aligned} \tag{7.63}$$

$$\begin{aligned}\Delta^{-1}F_{mnpq}F^{mnpq}\Big|_\star = \frac{16m_7^2\Delta^6|_\star}{27}\Big(&14\mathcal{X}^4+35\mathcal{X}^3\mathcal{Z}+178\mathcal{X}^2\mathcal{Z}^2+35\mathcal{X}\mathcal{Z}^3+14\mathcal{Z}^4\\ &-189(3\mathcal{X}-\mathcal{Z})(3\mathcal{Z}-\mathcal{X})\Big),\\ &+3\sqrt{5}(\mathcal{X}+\mathcal{Z})(19\mathcal{X}^2-50\mathcal{X}\mathcal{Z}+19\mathcal{Z}^2).\end{aligned} \tag{7.64}$$

Plugging these expressions into Eqs. (5.19, 5.31) then verifies the external Einstein equations.

Finally, one must verify the internal Einstein equations. Unfortunately, the required terms like $\mathring{D}_p\tilde{\Gamma}^p_{mn}\Big|_\star$ and $\tilde{\Gamma}^q_{mp}\tilde{\Gamma}^p_{qn}\Big|_\star$ are too long to actually display them here. Only the complete Ricci tensor becomes manageable, since a lot of terms cancel. As turns out, it is even more convenient to raise one index in Eq. (5.20) with $\Delta^{-1}g^{mn}(x,y)$. Therefore, the expressions for $\Delta^{-1}R_m{}^n(x,y)$ and $\Delta^{-1}F_{mpqr}F^{npqr}(x,y)$ are computed here. Using Eqs. (7.22 – 7.24) as well as the S^7 tensor identities in Appendix C in the definition of the internal Ricci tensor (Eq. (5.32)), one finally finds

$$\begin{aligned}\Delta^{-1}R_m{}^n\Big|_\star = \frac{m_7^2\Delta^6|_\star}{36^2}\Big(&r_0(\mathcal{X},\mathcal{Z})\delta^n_m + r_1(\mathcal{X},\mathcal{Z})\xi_m{}^n + r_1(\mathcal{Z},\mathcal{X})\zeta_m{}^n + r_2(\mathcal{X},\mathcal{Z})F_m{}^n\\ &+r_3(\mathcal{X},\mathcal{Z})\xi_m\xi^n + r_3(\mathcal{Z},\mathcal{X})\zeta_m\zeta^n + r_4(\mathcal{X},\mathcal{Z})F_mF^n\\ &+r_5(\mathcal{X},\mathcal{Z})\xi_m\zeta^n + r_5(\mathcal{Z},\mathcal{X})\zeta_m\xi^n + r_6(\mathcal{X},\mathcal{Z})\xi_mF^n\\ &-r_6(\mathcal{Z},\mathcal{X})\zeta_mF^n + r_7(\mathcal{X},\mathcal{Z})F_m\xi^n - r_7(\mathcal{Z},\mathcal{X})F_m\zeta^n\Big),\end{aligned} \tag{7.65}$$

where

$$
\begin{aligned}
r_0(\mathcal{X},\mathcal{Z}) &= \frac{2\sqrt{5}}{3}(\mathcal{X}+\mathcal{Z})\left(17\mathcal{X}^4-80\mathcal{X}^3\mathcal{Z}-66\mathcal{X}^2\mathcal{Z}^2-80\mathcal{X}\mathcal{Z}^3+17\mathcal{Z}^4\right)\\
&\quad+\frac{40}{3}\left(13\mathcal{X}^4-134\mathcal{X}^3\mathcal{Z}-214\mathcal{X}^2\mathcal{Z}^2-134\mathcal{X}\mathcal{Z}^3+13\mathcal{Z}^4\right)\\
&\quad+40\sqrt{5}(\mathcal{X}+\mathcal{Z})\left(17\mathcal{X}^2-58\mathcal{X}\mathcal{Z}+17\mathcal{Z}^2\right)-840(5\mathcal{X}^2+2\mathcal{X}\mathcal{Z}+5\mathcal{Z}^2),\\
r_1(\mathcal{X},\mathcal{Z}) &= -10080\sqrt{5}(\mathcal{X}^2-\mathcal{Z}^2)-96\left(41\mathcal{X}^3-45\mathcal{X}^2\mathcal{Z}-9\mathcal{X}\mathcal{Z}^2-35\mathcal{Z}^3\right)\\
&\quad-8\sqrt{5}(\mathcal{X}+\mathcal{Z})\left(17\mathcal{X}^3-55\mathcal{X}^2\mathcal{Z}-33\mathcal{X}\mathcal{Z}^2-25\mathcal{Z}^3\right),\\
r_2(\mathcal{X},\mathcal{Z}) &= 10080(\mathcal{X}^2-\mathcal{Z}^2)+96\sqrt{5}(\mathcal{X}-\mathcal{Z})\left(7\mathcal{X}^2+10\mathcal{X}\mathcal{Z}+7\mathcal{Z}^2\right)\\
&\quad+200(\mathcal{X}-\mathcal{Z})(\mathcal{X}+\mathcal{Z})^3,\\
r_3(\mathcal{X},\mathcal{Z}) &= 672\sqrt{5}(5\mathcal{X}+13\mathcal{Z})+32(45\mathcal{X}^2-160\mathcal{X}\mathcal{Z}+79\mathcal{Z}^2)\\
&\quad+\frac{8\sqrt{5}}{3}\left(17\mathcal{X}^3+43\mathcal{X}^2\mathcal{Z}-149\mathcal{X}\mathcal{Z}^2+17\mathcal{Z}^3\right),\\
r_4(\mathcal{X},\mathcal{Z}) &= -2016(\mathcal{X}^2+10\mathcal{X}\mathcal{Z}+\mathcal{Z}^2)-96\sqrt{5}(\mathcal{X}+\mathcal{Z})(\mathcal{X}^2-50\mathcal{X}\mathcal{Z}+\mathcal{Z}^2),\\
r_5(\mathcal{X},\mathcal{Z}) &= -672\sqrt{5}(13\mathcal{X}+5\mathcal{Z})-64(50\mathcal{X}^2-33\mathcal{X}\mathcal{Z}-5\mathcal{Z}^2)\\
&\quad-\frac{8\sqrt{5}}{3}\left(81\mathcal{X}^3-61\mathcal{X}^2\mathcal{Z}+75\mathcal{X}\mathcal{Z}^2+25\mathcal{Z}^3\right),\\
r_6(\mathcal{X},\mathcal{Z}) &= 336\sqrt{5}(\mathcal{X}^2+10\mathcal{X}\mathcal{Z}+\mathcal{Z}^2)+16\left(5\mathcal{X}^3-188\mathcal{X}^2\mathcal{Z}-175\mathcal{X}\mathcal{Z}^2-38\mathcal{Z}^3\right),\\
r_7(\mathcal{X},\mathcal{Z}) &= -4032(5\mathcal{X}+13\mathcal{Z})-48\sqrt{5}(35\mathcal{X}^2-118\mathcal{X}\mathcal{Z}+47\mathcal{Z}^2)\\
&\quad-16\left(25\mathcal{X}^3+116\mathcal{X}^2\mathcal{Z}-75\mathcal{X}\mathcal{Z}^2+66\mathcal{Z}^3\right).
\end{aligned}
$$

In the same way, one computes

$$
\begin{aligned}
&\Delta^{-1}F_{mpqr}F^{npqr}\Big|_\star \\
&\quad= -6\Delta^{-1}R_m{}^n+\frac{4\,m_7^2\,\Delta^6|_\star}{81}\Bigg(14\mathcal{X}^4+35\mathcal{X}^3\mathcal{Z}+178\mathcal{X}^2\mathcal{Z}^2+35\mathcal{X}\mathcal{Z}^3+14\mathcal{Z}^4\\
&\qquad+3\sqrt{5}(\mathcal{X}+\mathcal{Z})(19\mathcal{X}^2-50\mathcal{X}\mathcal{Z}+19\mathcal{Z}^2)+\frac{63}{4}(29\mathcal{X}^2-190\mathcal{X}\mathcal{Z}+29\mathcal{Z}^2)\Bigg)\,\delta_m^n
\end{aligned}
\tag{7.66}
$$

from Eqs. (7.50, 7.61). Together with Eqs. (7.58, 7.64), the internal Einstein equations are finally verified.

8. Conclusions

This thesis presents the complete bosonic embedding of gauged $N = 8$ supergravity into 11 dimensions. The higher-dimensional fields are redefined in a non-linear way, such that their supersymmetry transformations are SU(8) and E_7 covariant. Only these non-linear reformulations can be related to the fields of $N = 8$ supergravity. This is the basis for finding explicit embedding formulae for the 11-dimensional fields in terms of the four-dimensional ones. As it turns out, the vector uplift Ansätze can simply be found when the scalar Ansätze are known.

The first part summarizes all scalar embedding formulae. It presents the well known Ansätze for the inverse metric [43], the internal three-form with mixed index structure [45] and the internal six-form potential [47]. Furthermore, it derives new *direct* uplift formulae for the metric, the warp factor and the full three-form potential [49, 50]. However, the new Ansatz for the full internal three-form flux does not reveal its full antisymmetry. This may be a hint that one can further simplify the expression for A_{mnp} in Eq. (4.49) using some E_7 identities for the four-dimensional scalar fields.

The presented Ansätze are sufficient to construct a complete bosonic solution of 11-dimensional supergravity in terms of the four-dimensional fields of $N = 8$ supergravity. Such a solution may further be simplified by restricting the four-dimensional spacetime to be maximally symmetric and evaluating the fields at a stationary point of the scalar potential. Within the resulting Freund-Rubin solution, the 11-dimensional vector fields must vanish and the given embedding formulae are complete. The consistency checks of such solutions then require the calculation of the internal four-form field-strength and the Freund-Rubin term. Here, the corresponding uplift Ansätze are also derived [48, 49, 50].

The second part of this thesis discusses group invariant solutions of 11-dimensional supergravity. These are obtained when uplifting certain gaugings of $N = 8$ supergravity, which are related to different deformations of the seven-sphere. In such cases, the embedding formulae further simplify and the resulting 11-dimensional fields can be written in terms of certain group invariant tensors that are adapted to the deformed seven-sphere.

In particular, the methods are used to embed two different gaugings of $N = 8$ supergravity into 11 dimensions. The first example is the rederivation of the complete G_2 invariant Freund-Rubin solution [43, 48, 60]. Secondly, the full bosonic uplift of SO(3)×SO(3) gauged supergravity is presented [51]. Finally, the consistency of these solutions is explicitly verified.

In future, one may also find direct uplift Ansätze for the 11-dimensional Riemann and Ricci tensor components, which are given by the second derivatives of the metric.

Hence, one could find new simple expressions in full analogy to the derivation of the field-strength Ansatz.

Furthermore, *flow solutions* are rarely investigated so far [58]. These are complete group invariant solutions of 11-dimensional supergravity, in which the coordinates on the scalar manifold $\lambda^{(r)}(x)$, $\mu^{(s)}(x)$ *flow* from one stationary point to another in spacetime. This must be consistent with the equations of motion, which reduce to certain differential equations for $\lambda^{(r)}(x)$ and $\mu^{(s)}(x)$. A first example could be a flow from the $\mathrm{SO}(7)^+$ invariant stationary point to the $\mathrm{SO}(7)^-$ one (the common subgroup is G_2). Both regions have interesting features: the $\mathrm{SO}(7)^+$ invariant point is described by a vanishing three-form potential and a non-trivial internal metric, whereas the $\mathrm{SO}(7)^-$ invariant solution comprises a non-trivial flux.

Finally, this thesis only discusses the compact gaugings of $N = 8$ supergravity, which are derived via the S^7 reduction of 11-dimensional supergravity. However, the methods provided here could also be applied for other truncations. As a first example, one may extend the theory to the non-compact $\mathrm{CSO}(p, q, r)$ gaugings [65, 66]. In this case, the $I, J, \ldots$ indices of the Killing forms are raised and lowered with the $\mathrm{CSO}(p, q, r)$-metric η_{IJ} instead of the $\mathrm{SO}(8)$ metric δ_{IJ}. This effects the definition of the matrix $\mathcal{R}^{\mathcal{M}}{}_{\mathcal{N}}$ in Eqs. (4.18 – 4.20). In particular, the scalar embedding formulae will be slightly modified. However, the subsequent Ansätze for the four-form field-strength and the Freund-Rubin term will change more dramatically: Eqs. (4.52 – 4.55) do not hold if the $I, J, \ldots$ indices of the Killing forms are raised and lowered with the full $\mathrm{CSO}(p, q, r)$ metric. Since the Ansätze depend on those identities, it will take much more effort to derive adapted Ansätze within the non-compact gaugings. Finally, the presented methods may also be used for the reduction from type IIB supergravity to five dimensions [67, 68, 69].

A. Useful Identities for Gamma Matrices and Killing Forms

This Appendix summarizes some useful identities for the antisymmetrized products of Γ and $\mathring{\Gamma}$ matrices that are defined in Eqs. (3.18, 4.10). In particular, these relations can also be translated for the Killing forms that are defined in Eqs. (4.9).

First of all, Γ matrices with one and two indices are antisymmetric and Γ matrices with three indices are symmetric. In particular, the two sets

$$\left(\mathbb{I}_{8\times 8},\quad \Gamma_m,\quad \Gamma_{mn},\quad \Gamma_{mnp}\right), \qquad \left(\mathbb{I}_{8\times 8},\quad \mathring{\Gamma}_m,\quad \mathring{\Gamma}_{mn},\quad \mathring{\Gamma}_{mnp}\right)$$

each contain $1 + 7 + 21 + 35 = 64$ independent matrices — they both span the vector space of 8×8 matrices. In these bases,

$$\begin{aligned}
\Gamma_{m_1\ldots m_7} &= -i\epsilon_{m_1\ldots m_7}\mathbb{I}_{8\times 8}, & \mathring{\Gamma}_{m_1\ldots m_7} &= -i\mathring{\eta}_{m_1\ldots m_7}\mathbb{I}_{8\times 8},\\
\Gamma_{m_1\ldots m_6} &= -i\epsilon_{m_1\ldots m_7}\Gamma^{m_7}, & \mathring{\Gamma}_{m_1\ldots m_6} &= -i\mathring{\eta}_{m_1\ldots m_7}\mathring{\Gamma}^{m_7},\\
\Gamma_{m_1\ldots m_5} &= \frac{i}{2}\epsilon_{m_1\ldots m_7}\Gamma^{m_6 m_7}, & \mathring{\Gamma}_{m_1\ldots m_5} &= \frac{i}{2}\mathring{\eta}_{m_1\ldots m_7}\mathring{\Gamma}^{m_6 m_7},\\
\Gamma_{m_1\ldots m_4} &= \frac{i}{3!}\epsilon_{m_1\ldots m_7}\Gamma^{m_5\cdots m_7}, & \mathring{\Gamma}_{m_1\ldots m_4} &= \frac{i}{3!}\mathring{\eta}_{m_1\ldots m_7}\mathring{\Gamma}^{m_5\cdots m_7}.
\end{aligned}$$

Secondly, using the Clifford algebra in Eqs. (3.17, 4.8), the Γ matrices satisfy the useful contraction relations

$$\mathrm{Tr}\left(\Gamma^m\Gamma^n\right) = 8g^{mn}, \qquad \mathrm{Tr}\left(\Gamma^m\Gamma^{np}\right) = 0, \qquad \mathrm{Tr}\left(\Gamma^{mn}\Gamma_{pq}\right) = -16\delta^{mn}_{pq}, \tag{A.1}$$

$$\mathrm{Tr}\left(\mathring{\Gamma}^m\mathring{\Gamma}^n\right) = 8\mathring{g}^{mn}, \qquad \mathrm{Tr}\left(\mathring{\Gamma}^m\mathring{\Gamma}^{np}\right) = 0, \qquad \mathrm{Tr}\left(\mathring{\Gamma}^{mn}\mathring{\Gamma}_{pq}\right) = -16\delta^{mn}_{pq}. \tag{A.2}$$

These can be translated to relations for the Killing forms in Eq. (4.9), using the orthonormality of the Killing spinors in Eq. (4.6):

$$K_m{}^{IJ}K_n{}^{IJ} = 8\mathring{g}_{mn}, \qquad K_m{}^{IJ}K_{np}{}^{IJ} = 0, \qquad K^{mn\,IJ}K_{pq}{}^{IJ} = 16\delta^{mn}_{pq}. \tag{A.3}$$

Similar identities can also be obtained for the traces of *any* products of Γ matrices or Killing forms, e.g.

$$\mathrm{Tr}\left(\Gamma^m\Gamma^n\Gamma_{pq}\right), \qquad \mathrm{Tr}\left(\mathring{\Gamma}_{mnp}\mathring{\Gamma}^q\mathring{\Gamma}^{rs}\mathring{\Gamma}^t\right), \qquad K_m{}^{IJ}K_n{}^{JK}K^{pq\,KI}.$$

Indeed, using the Clifford algebra and the definition of the antisymmetrized products of Γ matrices in Eqs. (3.17, 3.18, 4.8, 4.10), these expressions can be written solely in terms of the respective metric and Levi-Cevita tensor. The explicit relations are not listed here, as they are directly used within a `FORM` program.

Furthermore, the following bi-linears in the Γ matrices represent a basis of selfdual and anti-selfdual SU(8) tensors on the deformed seven-sphere [55]:

$$\begin{aligned} \text{selfdual}: \quad & \Gamma_{m\,[AB}\Gamma^{m}{}_{CD]}, \qquad \Gamma_{mn\,[AB}\Gamma^{n}{}_{CD]}, \qquad \Gamma^{m}{}_{[AB}\Gamma^{n}{}_{CD]}\,; \\ \text{anti}-\text{selfdual}: \quad & \Gamma^{[mn}{}_{[AB}\Gamma^{p]}{}_{CD]}. \end{aligned} \tag{A.4}$$

The same holds for bi-linears in the $\mathring{\Gamma}$ matrices. Hence, the following combinations of the Killing forms represent a basis of selfdual and anti-selfdual SO(8) tensors on the deformed seven-sphere:

$$\begin{aligned} \text{selfdual}: \quad & K_{m}{}^{[IJ}K^{m\,KL]}, \qquad K_{mn}{}^{[IJ}K^{n\,KL]}, \qquad K_{m}{}^{[IJ}K_{n}{}^{KL]}\,; \\ \text{anti}-\text{selfdual}: \quad & K_{[mn}{}^{[IJ}K_{p]}{}^{KL]}. \end{aligned} \tag{A.5}$$

These two bases in Eqs. (A.4, A.5) are in some sense 'orthogonal', i.e. one has

$$\Gamma^{m}{}_{[AB}\Gamma^{n}{}_{CD]}\Gamma^{p}{}_{AB}\Gamma^{q}{}_{CD} = 16 g^{(mn}g^{pq)}, \qquad K_{m}{}^{[IJ}K_{n}{}^{KL]}K_{p}{}^{IJ}K_{q}{}^{KL} = 16\mathring{g}_{(mn}\mathring{g}_{pq)}, \tag{A.6}$$

$$\Gamma_{mp\,[AB}\Gamma^{p}{}_{CD]}\Gamma_{nq\,AB}\Gamma^{q}{}_{CD} = -192 g_{mn}, \qquad K_{mp}{}^{[IJ}K^{p\,KL]}K_{nq}{}^{IJ}K^{q\,KL} = 192\mathring{g}_{mn}, \tag{A.7}$$

$$\Gamma^{[mn}{}_{[AB}\Gamma^{p]}{}_{CD]}\Gamma_{[qr\,AB}\Gamma_{s]CD} = -32\delta^{mnp}_{qrs}, \qquad K_{[qr}{}^{[IJ}K_{s]}{}^{KL]}K^{[mn\,IJ}K^{p]KL} = 32\delta^{mnp}_{qrs}, \tag{A.8}$$

whereas all other contractions, such as

$$\Gamma^{m}{}_{[AB}\Gamma^{n}{}_{CD]}\Gamma_{pq\,AB}\Gamma^{q}{}_{CD} = 0, \qquad K_{m}{}^{[IJ}K_{n}{}^{KL]}K_{pq}{}^{IJ}K^{q\,KL} = 0 \tag{A.9}$$

vanish identically.

Related to these bases are the following relations [55, 60]:

$$\Gamma_{m\,AB}\Gamma^{m}{}_{CD} + 2\delta^{AB}_{CD} = \Gamma_{m\,[AB}\Gamma^{m}{}_{CD]}, \tag{A.10}$$

$$\Gamma_{mn\,AB}\Gamma^{n}{}_{CD} + \Gamma_{mn\,CD}\Gamma^{n}{}_{AB} = 2\Gamma_{mn\,[AB}\Gamma^{n}{}_{CD]}, \tag{A.11}$$

$$\Gamma_{mn\,AB}\Gamma^{n}{}_{CD} - \Gamma_{mn\,CD}\Gamma^{n}{}_{AB} = -4\left(\delta_{C[A}\Gamma_{m\,B]D} - \delta_{D[A}\Gamma_{m\,B]C}\right), \tag{A.12}$$

$$\Gamma^{mn}{}_{[AB}\Gamma^{p}{}_{CD]} = -\frac{1}{3}g^{p[m}\Gamma^{n]q}{}_{[AB}\Gamma_{q\,CD]} + \Gamma^{[mn}{}_{[AB}\Gamma^{p]}{}_{CD]}, \tag{A.13}$$

$$\begin{aligned} \Gamma^{mn}{}_{[AB}\Gamma^{pq}{}_{CD]} = & -2g^{m[p}\Gamma^{q]}{}_{[AB}\Gamma^{n}{}_{CD]} + 2g^{n[p}\Gamma^{q]}{}_{[AB}\Gamma^{m}{}_{CD]} \\ & + \frac{2}{3}g^{m[p}g^{q]n}\Gamma_{r\,[AB}\Gamma^{r}{}_{CD]} + \Gamma^{[mn}{}_{[AB}\Gamma^{pq]}{}_{CD]}, \end{aligned} \tag{A.14}$$

as well as

$$K_m{}^{IJ}K^{m\,KL} - 2\delta^{IJ}_{KL} = K_m{}^{[IJ}K^{m\,KL]}, \tag{A.15}$$

$$K_{mn}{}^{IJ}K^{nKL} + K_{mn}{}^{KL}K^{nIJ} = 2K_{mn}{}^{[IJ}K^{nKL]}, \tag{A.16}$$

$$K_{mn}{}^{IJ}K^{nKL} - K_{mn}{}^{KL}K^{nIJ} = -8\delta^{[I}{}_{[K}K_m{}^{J]}{}_{L]}, \tag{A.17}$$

$$K_{mn}{}^{[IJ}K_p{}^{KL]} = -\frac{1}{3}\mathring{g}_{p[m}K_{n]q}{}^{[IJ}K^{q\,KL]} + K_{[mn}{}^{[IJ}K_{p]}{}^{KL]}, \tag{A.18}$$

$$\begin{aligned} K_{mn}{}^{[IJ}K_{pq}{}^{KL]} &= 2\mathring{g}_{m[p}K_{q]}{}^{[IJ}K_n{}^{KL]} - 2\mathring{g}_{n[p}K_{q]}{}^{[IJ}K_m{}^{KL]} \\ &\quad - \frac{2}{3}\mathring{g}_{m[p}\mathring{g}_{q]n}K_r{}^{[IJ}K^{r\,KL]} + K_{[mn}{}^{[IJ}K_{pq]}{}^{KL]}. \end{aligned} \tag{A.19}$$

Finally, it is convenient to define the selfdual tensor

$$K^{IJKL} = K_m{}^{[IJ}K^{m\,KL]}, \tag{A.20}$$

which satisfies the convenient identities [48]

$$K^{IJKP}K_{LMNP} = 6\delta^{IJK}_{LMN} + 9\delta^{[I}{}_{[L}K^{JK]}{}_{MN]}, \tag{A.21}$$

$$K^{[IJKL}K^{M]NPQ} = \frac{1}{5}\epsilon^{IJKLMNPQ} + 12K^{[IJK}{}_{[N}\delta^L{}_P\delta^{M]}{}_{Q]}, \tag{A.22}$$

$$\begin{aligned} K^{m\,IJ}K^{n\,KL}K_{mn}{}^{MN} &= 8\delta^{[I}{}_{[K}\delta^{J][M}\delta^{N]}{}_{L]} \\ &\quad + 4\delta^{[M}{}_{[I}K^{N]}{}_{J]KL} + 4\delta^{[K}{}_{[M}K^{L]}{}_{N]IJ} - 4\delta^{[I}{}_{[K}K^{J]}{}_{L]MN}. \end{aligned} \tag{A.23}$$

B. Factorization of the $\mathcal{C}$ Tensor

This appendix shows that one can extract a Kronecker-delta out of the $\mathcal{C}$ tensor defined in Eq. (4.47) [49],

$$\mathcal{C}_{pq}{}^{ijkl}(x,y) = \frac{4}{3}\delta^{[i}{}_{[p}\left(\mathcal{C}_{1\,q]}{}^{jkl]}(x,y) + 2\mathcal{C}_{2\,q]}{}^{jkl]}(x,y) - 2T_{q]}{}^{jkl]}(x)\right), \tag{B.1}$$

where

$$\mathcal{C}_{1\,p}{}^{ijk}(x,y) = K^{IJKL}(y)\left(u^{jk}{}_{IJ} + v^{jk\,IJ}\right)\left(u^{im}{}_{KM}u_{pm}{}^{LM} - v^{im\,KM}v_{pm\,LM}\right)(x), \tag{B.2}$$

$$\begin{aligned}\mathcal{C}_{2\,p}{}^{ijk}(x,y) = K^{IJKL}(y)\left(u^{jk}{}_{IM} + v^{jk\,IM}\right)\Big[&\left(u^{im}{}_{[JK}v_{pm\,LM]} - v^{im\,[JK}u_{pm}{}^{LM]}\right) \\ &- \frac{1}{8}\delta^i_p\left(u^{mn}{}_{[JK}v_{mn\,LM]} - v^{mn\,[JK}u_{mn}{}^{LM]}\right)\Big](x).\end{aligned} \tag{B.3}$$

The selfdual tensor K^{IJKL} is defined as a certain combination of Killing vectors in Eq. (A.20) and satisfies some useful relations given in Appendix A. The third term in Eq. (B.1) represents the T tensor, which is defined in Eq. (2.11). Note that the only difference between $C_{1\,p}{}^{ijk}$ and the T tensor is the K^{IJKL} factor in Eq. (B.2) instead of a δ^{IJ}_{KL} factor in Eq. (2.11). This gives rise to interpret $\mathcal{C}_1$ and $\mathcal{C}_2$ as the *y-dependent twins of the T tensor.*

In order to prove Eq. (B.1), one starts with Eq. (4.47) and replaces the tensors $\mathcal{A}_m{}^{ijkl}$ and $\mathcal{B}_m{}^{ijkl}$ with the respective expressions in Eqs. (4.43, 4.44). Secondly, using Eqs. (A.15, A.23) gives

$$\begin{aligned}\mathcal{C}_{pq}{}^{ijkl} = &-2K^{IJKL}\left(u_{pq}{}^{IM} + v_{pq\,IM}\right)\left(u^{ij}{}_{[JK}u^{kl}{}_{LM]} - v^{ij\,[JK}v^{kl\,LM]}\right) \\ &- K^{IJKL}\left(u_{pq}{}^{KL} + v_{pq\,KL}\right)\left(u^{ij}{}_{IM}v^{kl\,JM} - v^{ij\,IM}u^{kl}{}_{JM}\right) - \frac{8}{3}\delta^{[i}{}_{[p}T_{q]}{}^{jkl]},\end{aligned} \tag{B.4}$$

which can be rearranged,

$$\begin{aligned}\mathcal{C}_{pq}{}^{ijkl} = &\,2K^{IJKL}\left(u^{[ij}{}_{IJ} + v^{[ij\,IJ}\right)\left(u^{kl]}{}_{KM}u_{pq}{}^{LM} - v^{kl]\,KM}v_{pq\,LM}\right) \\ &+ 4K^{IJKL}\left(u^{[ij}{}_{IM} + v^{[ij\,IM}\right)\left(u^{kl]}{}_{[JK}v_{pq\,LM]} - v^{kl]\,[JK}u_{pq}{}^{LM]}\right) - \frac{8}{3}\delta^{[i}{}_{[p}T_{q]}{}^{jkl]}.\end{aligned} \tag{B.5}$$

Finally, using Eqs. (2.31, 2.32) completes the proof of Eq. (B.1). In order to keep the formulae short, this factorization of the $\mathcal{C}_{pq}{}^{ijkl}$ tensor is not inserted into the uplift

Ansätze for the warp factor and the three-form in Eqs. (4.46, 4.49). However, one should always keep in mind that these expressions can still be simplified via Eq. (B.1). For example, this could be a first step in proving the explicit antisymmetry of the three-form Ansatz.

C. Useful Identities for the SO(3)×SO(3) Invariant S^7 Tensors

This section gives all identities for the S^7 tensors defined in Eqs. (7.19 – 7.21), which are required to simplify the obtained 11-dimensional SO(3)×SO(3) invariant fields. The starting point to derive such identities are the relations between the SO(8) tensors $Y^\pm$ and $Z^\pm$ in Eqs. (7.4 – 7.8). All these relations can be contracted with the following combinations of Killing forms

$$K_m{}^{IJ} K_n{}^{KL}, \qquad K_{mn}{}^{IJ} K_p{}^{KL}, \qquad K_m{}^{IJ} K_{np}{}^{KL}, \qquad K_{mn}{}^{IJ} K_{pq}{}^{KL}.$$

The contraction with each of these terms gives another identity when using Eqs. (7.25 – 7.32) and the relations in Eq. (5.11).

The identities are listed in Tables C.1 – C.7. Each table represents identities that are derived from a different relation between the SO(8) tensors. The only used cubic identities are given in Eq. (7.8). The reason is that the identities that follow from Eqs. (7.9, 7.10) can also be obtained from Eqs. (7.4 – 7.7).

Finally, one finds the useful relations involving the F_{IJ} tensor from Eqs. (7.1, 7.2):

$$Y^+_{IKLM} Z^+_{JKLM} = Z^-_{IKLM} Y^-_{JKLM} = 12 F_{IJ}, \tag{C.1}$$

$$Y^\pm_{IJKL} F^{KL} = Z^\pm_{IJKL} F^{KL} = 0, \tag{C.2}$$

$$8 Y^\pm_{[IJK|M|} F^M{}_{L]} = \pm Z^\pm_{IJKL}, \qquad 8 Z^\pm_{[IJK|M|} F^M{}_{L]} = \mp Y^\pm_{IJKL}. \tag{C.3}$$

These can also be contracted with the Killing forms in order to derive useful identities for the SO(3)×SO(3) invariant S^7 tensors. Here, one must use the explicit decomposition of the $Y^\pm$ and $Z^\pm$ tensors in Eqs. (7.17, 7.18) as well as the contraction relations for the Killing forms described in Appendix A.

Note that all derived identities are presented in a systematic way — the derivation of later identities may require previous relations.

Table C.1.: Identities derived from Eq. (7.4) and Eq. (7.8).

(i)	$\xi^{mn}\xi_{mn} = \frac{3}{2} + \frac{\xi^2}{6}, \quad \xi^p\xi_p = 9 - \xi^2,$
(ii)	$\zeta^{mn}\zeta_{mn} = \frac{3}{2} + \frac{\zeta^2}{6}, \quad \zeta^p\zeta_p = 9 - \zeta^2$
(iii)	$S_{mnp}S^{mnp} = 6, \quad \xi^{mn}\xi_n = 0,$
(iv)	$T_{mnp}T^{mnp} = 6 \quad \zeta^{mn}\zeta_n = 0$
(v)	$\xi^{mp}\xi^n{}_p = \left(\frac{1}{4} - \frac{\xi^2}{36}\right)\mathring{g}^{mn} + \frac{\xi}{3}\xi^{mn} - \frac{1}{36}\xi^m\xi^n,$
(vi)	$\zeta^{mp}\zeta^n{}_p = \left(\frac{1}{4} - \frac{\zeta^2}{36}\right)\mathring{g}^{mn} + \frac{\zeta}{3}\zeta^{mn} - \frac{1}{36}\zeta^m\zeta^n$
(vii)	$S^{mpq}S^n{}_{pq} = \left(1 - \frac{\zeta^2}{9}\right)\mathring{g}^{mn} - \frac{1}{9}\zeta^m\zeta^n + \frac{2\zeta}{3}\zeta^{mn},$
(viii)	$T^{mpq}T^n{}_{pq} = \left(1 - \frac{\xi^2}{9}\right)\mathring{g}^{mn} - \frac{1}{9}\xi^m\xi^n + \frac{2\xi}{3}\xi^{mn}$
(ix)	$\mathring{\eta}_{mnqrstu}T^{qrs}T^{tu}{}_p = 8\xi_{[m}\xi_{n]p} - \frac{4}{3}\xi\xi_{[m}\mathring{g}_{n]p},$
(x)	$\mathring{\eta}_{mnqrstu}S^{qrs}S^{tu}{}_p = 8\zeta_{[m}\zeta_{n]p} - \frac{4}{3}\zeta\zeta_{[m}\mathring{g}_{n]p}$
(xi)	$S^{mnr}S_{pqr} = 2\zeta^{[m}{}_{[p}\zeta^{n]}{}_{q]} + \left(\frac{1}{2} - \frac{\zeta^2}{18}\right)\delta^{mn}_{pq} - \frac{1}{9}\zeta^{[m}\zeta_{[p}\delta_{q]}{}^{n]}$
(xii)	$T^{mnr}T_{pqr} = 2\xi^{[m}{}_{[p}\xi^{n]}{}_{q]} + \left(\frac{1}{2} - \frac{\xi^2}{18}\right)\delta^{mn}_{pq} - \frac{1}{9}\xi^{[m}\xi_{[p}\delta_{q]}{}^{n]}$

Table C.2.: Identities derived from Eq. (7.5) and Eq. (C.1).

(i)	$\xi^m \zeta_m = -\xi\zeta, \qquad \xi^{mn}\zeta_{mn} = \frac{1}{6}\xi\zeta, \qquad S_{mnp}T^{mnp} = 0$
(ii)	$\mathring{\eta}^{mnpqrst} S_{npq} T_{rst} = 18 F^m$
(iii)	$\xi^{mn}\zeta_n = \frac{\xi}{6}\zeta^m - \frac{\zeta}{6}\xi^m + \frac{3}{2}F^m, \quad \zeta^{mn}\xi_n = -\frac{\xi}{6}\zeta^m + \frac{\zeta}{6}\xi^m - \frac{3}{2}F^m$
(iv)	$\xi^{mp}\zeta^n{}_p = -\frac{1}{36}\xi\zeta\mathring{g}^{mn} - \frac{1}{36}\xi^m\zeta^n + \frac{1}{6}\left(\zeta\xi^{mn} + \xi\zeta^{mn}\right) + \frac{1}{4}F^{mn}$
(v)	$S^m{}_{pq}T^{npq} = -\frac{1}{9}\xi\zeta\mathring{g}^{mn} - \frac{1}{18}\left(\xi^m\zeta^n + \zeta^m\xi^n\right) + \frac{1}{3}\left(\zeta\xi^{mn} + \xi\zeta^{mn}\right) - \frac{1}{2}F^{mn}$
(vi)	$\mathring{\eta}_{npqrstu} S_m{}^{qr} T^{stu} = -4\xi_{m[n}\zeta_{p]} - 4\zeta_{m[n}\xi_{p]} + \frac{2}{3}\zeta\mathring{g}_{m[n}\xi_{p]} + \frac{2}{3}\xi\mathring{g}_{m[n}\zeta_{p]} - 6\mathring{g}_{m[n}F_{p]}$
(vii)	$\mathring{\eta}_{npqrstu} T_m{}^{qr} S^{stu} = -4\xi_{m[n}\zeta_{p]} - 4\zeta_{m[n}\xi_{p]} + \frac{2}{3}\zeta\mathring{g}_{m[n}\xi_{p]} + \frac{2}{3}\xi\mathring{g}_{m[n}\zeta_{p]} + 6\mathring{g}_{m[n}F_{p]}$
(viii)	$S^{mnr}T_{pqr} + T^{mnr}S_{pqr} = -\frac{1}{9}\xi\zeta\delta^{mn}_{pq} - \frac{1}{9}\xi^{[m}\zeta_{[p}\delta^{n]}_{q]} - \frac{1}{9}\zeta^{[m}\xi_{[p}\delta^{n]}_{q]} + 4\xi^{[m}{}_{[p}\zeta^{n]}{}_{q]}$

Table C.3.: Identities derived from Eq. (7.6) and Eq. (7.7).

(i)	$S_{mnp}\xi^p + T_{mnp}\zeta^p = 0, \qquad S_{mnp}\zeta^p = T_{mnp}\xi^p = 0$
(ii)	$S_{qmn}\zeta_p{}^q = S_{q[mn}\zeta_{p]}{}^q = \frac{\zeta}{3}S_{mnp} - \frac{1}{36}\mathring{\eta}_{mnpqrst}\zeta^q S^{rst}$
(iii)	$T_{qmn}\xi_p{}^q = T_{q[mn}\xi_{p]}{}^q = \frac{\xi}{3}T_{mnp} - \frac{1}{36}\mathring{\eta}_{mnpqrst}\xi^q T^{rst}$
(iv)	$4\zeta^{qr}T_{rmn} - \frac{1}{9}\mathring{\eta}^q{}_{mnstuv}\zeta^s T^{tuv} = 8S^{sq}{}_{[m}\xi_{n]s} - \frac{4}{3}\xi S^q{}_{mn}$
(v)	$4\xi^{qr}S_{rmn} - \frac{1}{9}\mathring{\eta}^q{}_{mnstuv}\xi^s S^{tuv} = 8T^{sq}{}_{[m}\zeta_{n]s} - \frac{4}{3}\zeta T^q{}_{mn}$
(vi)	$\xi^r{}_{[m}\mathring{\eta}_{n]pqrstu}S^{stu} - \zeta^r{}_{[p}\mathring{\eta}_{q]mnrstu}T^{stu} =$
	$\xi_{[m}S_{n]pq} - \zeta_{[p}T_{q]mn} + \frac{1}{6}\mathring{\eta}_{mnpqrst}(\zeta T^{rst} - \xi S^{rst})$

Table C.4.: Identities derived from Eq. (C.2) and Eq. (C.3).

(i)	$S^{mnp}F_{np} = 0, \quad S^{mnp}F_p = \frac{1}{12}\mathring{\eta}^{mnpqrst}S_{pqr}F_{st}$
(ii)	$T^{mnp}F_{np} = 0, \quad T^{mnp}F_p = \frac{1}{12}\mathring{\eta}^{mnpqrst}T_{pqr}F_{st}$
(iii)	$S^{q[mn}F^{p]}{}_q = \frac{2}{3}T^{mnp} + \frac{1}{18}\mathring{\eta}^{mnpqrst}S_{qrs}F_t$
(iv)	$T^{q[mn}F^{p]}{}_q = -\frac{2}{3}S^{mnp} + \frac{1}{18}\mathring{\eta}^{mnpqrst}T_{qrs}F_t$

Table C.5.: F tensor identities derived by contractions of the equations (iii) and (iv) in Table C.2 with ξ_m, ζ_m, F_m, ξ_{mq}, ζ_{mq} and F_{mq}.

(i)	$F_m\xi^m = \zeta, \quad F_m\zeta^m = -\xi, \quad F_m\xi^{mn} = \frac{1}{6}\zeta^n + \frac{\xi}{6}F^n, \quad F_m\zeta^{mn} = -\frac{1}{6}\xi^n + \frac{\zeta}{6}F^n$
(ii)	$F_{mn}\xi^n = -\zeta_m - \xi F_m, \quad F_{mp}\xi^p{}_n = \frac{\zeta}{6}\mathring{g}_{mn} - \frac{1}{6}F_m\xi_n - \zeta_{mn} + \frac{\xi}{6}F_{mn}$
(iii)	$F_{mn}\zeta^n = \xi_m - \zeta F_m, \quad F_{mp}\zeta^p{}_n = -\frac{\xi}{6}\mathring{g}_{mn} - \frac{1}{6}F_m\zeta_n + \xi_{mn} + \frac{\zeta}{6}F_{mn}$
(iv)	$F^mF_m = 1, \quad F^{mn}F_n = 0, \quad F^{mp}F_{pn} = F^mF_n - \delta^m_n$

Table C.6.: Subsequent identities derived by combining the equations (iv) and (v) in Table C.3.

(i)	$\xi_{sm}S_{np}{}^s + \zeta_{sm}T_{np}{}^s = \xi_{s[m}S_{np]}{}^s + \zeta_{s[m}T_{np]}{}^s$
(ii)	$\xi_{s[m}S_{np]}{}^s = \frac{1}{9}(\zeta T_{mnp} + 2\xi S_{mnp}) - \frac{1}{108}\mathring{\eta}_{mnpqrst}(2\zeta^qT^{rst} + \xi^qS^{rst})$
(iii)	$\zeta_{s[m}T_{np]}{}^s = \frac{1}{9}(\xi S_{mnp} + 2\zeta T_{mnp}) - \frac{1}{108}\mathring{\eta}_{mnpqrst}(2\xi^qS^{rst} + \zeta^qT^{rst})$

Table C.7.: Identities derived by contractions of the equations (ii) and (iii) in Table C.3 with ξ^p, ζ^p and F^p; and contractions of equations (iii) and (iv) in Table C.4 with ξ_m and ζ_m respectively.

(i)	$\mathring{\eta}_{mnpqrst}\xi^p\zeta^q S^{rst} = 6\zeta S_{mnp}\xi^p + 54 S_{mnp}F^p,$
(ii)	$\mathring{\eta}_{mnpqrst}\zeta^p\xi^q T^{rst} = 6\xi T_{mnp}\zeta^p - 54 T_{mnp}F^p$
(iii)	$\mathring{\eta}_{mnpqrst}F^p\zeta^q S^{rst} = 6 S_{mnp}\xi^p + 6\zeta S_{mnp}F^p,$
(iv)	$\mathring{\eta}_{mnpqrst}F^p\xi^q T^{rst} = -6 T_{mnp}\zeta^p + 6\xi T_{mnp}F^p$
(v)	$\mathring{\eta}_{mnpqrst}F^p\xi^q S^{rst} = 6\xi S_{mnp}F^p, \quad \mathring{\eta}_{mnpqrst}F^p\zeta^q T^{rst} = 6\zeta T_{mnp}F^p$

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Selbständigkeitserklärung

Ich erkläre, dass ich die Dissertation selbständig und nur unter Verwendung der von mir gemäß § 7 Abs. 3 der Promotionsordnung der Mathematisch-Naturwissenschaftlichen Fakultät, veröffentlicht im Amtlichen Mitteilungsblatt der Humboldt-Universität zu Berlin Nr. 126014 am 18.11.2014 angegebenen Hilfsmittel angefertigt habe.

Berlin, 13.09.2016 Olaf Krüger